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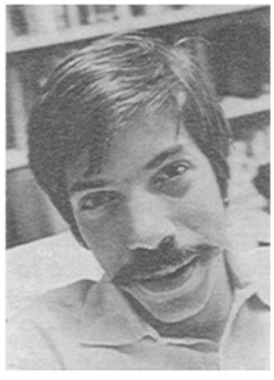
Recognizing safety and liveness*

Bowen Alpern[1] **and Fred B. Schneider**[2]

[1] IBM T.J. Watson Research Center, P.O. Box 218, Yorktown Heights, NY 10598, USA
[2] Department of Computer Science, Cornell University, Ithaca, NY 14853, USA

Bowen Alpern *was born in 1952. He received a Ph.D. in Computer Science from Cornell University in* 1986. *Currently, he is a Research Staff Member in the Mathematics Department of the IBM T.J. Watson Research Center.*



Fred B. Schneider *is an associate professor in the Computer Science Department at Cornell University. He received a Ph.D. in Computer Science from S.U.N.Y. at Stony Brook in* 1978 *and a B.S. from Cornell in* 1975. *Schneider is a member of the editorial boards of* Distributed Computing *and* Information Processing Letters. *He is also a member of the U.S. Army Committee on Recommendations for Basic Research, the College Board Committe for Advanced Placement Computer Science, IFIP Working Group 2.3 (Programming Methodology), and the Standing Organizing Committee for Principles of Distributed Computing Conferences. He has served on the program committee for PODC, POPL, SOSP, and FTCS.*

Abstract. A formal characterization for safety properties and liveness properties is given in terms of the structure of the Buchi automaton that specifies the property. The characterizations permit a property to be decomposed into a safety property and a liveness property whose conjunction is the original. The characterizations also give insight into techniques required to prove a large class of safety and liveness properties.

1 Introduction

Informally, a *safety property* stipulates that "bad things" do not happen during execution of a program and a *liveness property* stipulates that "good things" do happen (eventually) (Lamport 1977). Distinguishing between safety and liveness properties is useful because proving that a program satisfies a safety property involves an invariance argument while proving that a program satisfies a liveness property involves a well-foundedness argument. Thus, knowing whether a property is safety or liveness helps when deciding how to prove that the property holds.

The relationship between safety properties and invariance arguments and between liveness properties and well-foundedness arguments has, until now, not been formalized or proved. Rather, it was supported by practical experience in reasoning about concurrent and distributed programs in light of the informal definitions of safety and liveness given above. This paper substantiates that experience by formalizing safety and liveness in a way that permits the relationship between safety and invariance and between liveness and well-foundedness to be demonstrated for a large class of properties. In so doing, we give new characterizations of safety and liveness and prove that they satisfy the formal definitions in Alpern and Schneider (1985a).

Offprint requests to: F.B. Schneider

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We also give a method for decomposing properties into safety and liveness properties whose conjunction is the original.

We proceed as follows. Section 2 describes an automata-theoretic approach for specifying properties. Section 3 contains automata-theoretic characterizations of safety and liveness. Section 4 shows how a property can be expressed as the conjunction of a safety property and a liveness property. Section 5 discusses the relationship between safety and liveness and proof techniques. Section 6 discusses related work and Sect. 7 summarizes our contributions.

2 Histories and properties

A program π is assumed to be specified in terms of

- its set of atomic actions $\mathscr{A}_\pi$, and
- a predicate $Init_\pi$ that describes its possible initial states.

An execution of π can be viewed as an infinite sequence σ of program states

$$\sigma = s_0 s_1 \ldots,$$

which we call a *history*. State s_0 satisfies $Init_\pi$, and each following state results from executing a single enabled atomic action from $\mathscr{A}_\pi$ in the preceding state. For a terminating execution, an infinite sequence is obtained by repeating the final state. This corresponds to the view that a terminating execution is the same as non-terminating execution in which after some finite time – once the program has terminated – the state remains fixed.

A *property* is a set of infinite sequences of program states. We write $\sigma \models P$ to denote that (infinite sequence) σ is in property P. A program *satisfies* a property P if for each of its histories h, $h \models P$.

A property is usually specified by a characteristic predicate on sequences rather than by enumeration. Formulas of temporal logic can be interpreted as predicates on infinite sequences of states, and various formulations of temporal logic have been used for specifying properties (Lamport 1983; Lichtenstein et al. 1985; Manna and Pnueli 1981; Wolper 1983). However, for our purposes, it will be convenient to specify properties using Buchi automata – finite-state automata that accept infinite sequences (Vardi 1987). Mechanical procedures exist to translate linear-time and branching-time first-order temporal formulas into Buchi automata (Alpern 1986; Clarke et al. 1986; Wolper 1984), so using Buchi automata does not constitute a restriction.

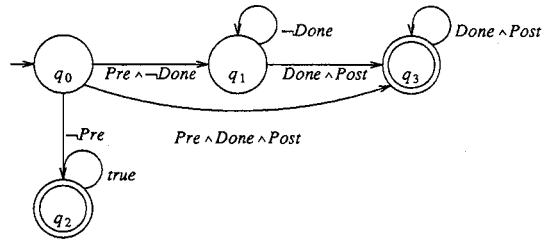

Fig. 1. m_{tc}

In fact, Buchi automata are more expressive than most temporal logic specification languages – there exist properties that can be specified using Buchi automata but cannot be specified in (standard) temporal logics (Wolper 1983).

A *Buchi automaton* (Eilenberg 1974) m accepts the sequences of program states that are in $L(m)$, the property it specifies. Figure 1 is a Buchi automaton m_{tc} that accepts (i) all infinite sequences in which the first state satisfies a predicate $\neg Pre$ and (ii) all infinite sequences consisting of a state satisfying *Pre*, followed by a (possibly empty) sequence of states satisfying $\neg Done$, followed by an infinite sequence of states satisfying $Done \wedge Post$. Thus, m_{tc} specifies *Total Correctness* with precondition *Pre* and postcondition *Post*, where *Done* holds if and only if the program has terminated.

Buchi automaton m_{tc} contains four *automaton states*, labeled q_0, q_1, q_2, and q_3. The *start state* (q_0) is denoted by an arc with no origin and *accepting states* (q_2 and q_3) by concentric circles. An infinite sequence is accepted by a Buchi automaton if and only if it causes the recognizer to be infinitely often in accepting states.

Arcs between automaton states are labeled by program-state predicates called *transition predicates*. These define transitions between automaton states based on the next symbol read from the input. For example, because there is an arc labeled $\neg Pre$ from q_0 to q_2 in m_{tc}, whenever m_{tc} is in q_0 and the next symbol read is a program state satisfying $\neg Pre$, then a transition to q_2 is made. If the next symbol read by a Buchi automaton satisfies no transition predicate on an arc emanating from the current automaton state, the input is rejected; in this case, we say the transition is *undefined* for that symbol. This is used in m_{tc} to ensure that an infinite sequence that starts with a state satisfying *Pre* ends in an infinite sequence of states that satisfy $Done \wedge Post$ – once m_{tc} enters q_3, every subsequent program state read must satisfy $Done \wedge Post$ or an undefined transition occurs.

We say that a Buchi automaton is *reduced* if from every state there is a path to an accepting

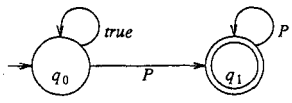

Fig. 2. m_{mono}

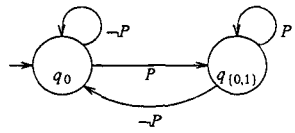

Fig. 3. m_{det}

state. Thus, m_{tc} is an example of a reduced Buchi automaton. Given an arbitrary Buchi automaton, an equivalent reduced Buchi automaton can always be obtained by deleting every state from which no accepting state is reachable.

When there is more than one start state or more than one transition is possible from some automaton state for some input symbol, the automaton is *non-deterministic*; otherwise it is *deterministic*. Thus, m_{tc} is deterministic because it has a single start state and because disjoint transition predicates label the arcs that emanate from each automaton state. Although any set of finite sequences recognizable by a non-deterministic (ordinary) finite-state automaton can be recognized by some deterministic (ordinary) finite-state automaton (Hopcroft and Ullman 1979), Buchi automata do not enjoy this equivalence. Some sets of infinite sequences can be recognized by non-deterministic Buchi automata but by not deterministic one (Eilenberg 1974). For example, consider m_{mono} of Fig. 2, which accepts all sequences with an infinite suffix of program states that satisfy P.[1] No deterministic Buchi automaton accepts this set of sequences. The standard subset construction for transforming a non-deterministic (ordinary) finite-state automaton to a deterministic one does not work when it is applied to $m_{mono}-m_{det}$ of Fig. 3 results, and no combination of q_0 and $q_{\{0,1\}}$ as accepting state causes m_{det} to accept the same set of sequences as m_{mono}.

Formally, a Buchi automaton m for a property of a program π is a five-tuple $(S, Q, Q_0, Q_\infty, \delta)$, where

S is the set of program states of π,

Q is the set of automaton states of m,

$Q_0 \subseteq Q$ is the set of start states of m,

$Q_\infty \subseteq Q$ is the set of accepting states of m,

$\delta \in (Q \times S) \to 2^Q$ is the *transition function* of m.

Transition predicates are derived from δ as follows. T_{ij}, the transition predicate associated with the arc from automaton state q_i to q_j, is the predicate that holds for all program states s such that $q_j \in \delta(q_i, s)$. Thus, T_{ij} is *false* if no symbol can cause a transition from q_i to q_j.

[1] In linear-time temporal logic, this property is characterized by $\Diamond\Box P$

In order to formalize when m accepts a sequence, some notation is required. For any sequence $\sigma = s_0 s_1 \ldots$,

$\sigma[i] = s_i$

$\sigma[..i] = s_0 s_1 \ldots s_i$

$\sigma[i..] = s_i s_{i+1} \ldots$

$|\sigma|$ = the length of σ (ω if σ is infinite).

Transition function δ can be extended to handle finite sequences of program states in the usual way:

$$\delta^*(q,\sigma) = \begin{cases} \{q\} & \text{if } |\sigma| = 0 \\ \{q' \mid (\exists q'' : q'' \in \delta^*(q, \sigma[..|\sigma|-2]) : \\ \quad q' \in \delta(q'', \sigma[|\sigma|-1]))\} & \text{if } 0 < |\sigma| < \omega. \end{cases}$$

A *run* of m for an infinite sequence σ is a sequence of automaton states that m could be in while reading σ. Thus, for ρ to be a run for σ, $\rho[0] \in Q_0$, and $(\forall i : 0 < i < |\sigma| : \rho[i] \in \delta(\rho[i-1], \sigma[i-1]))$. Let $\Gamma_m(\sigma)$ be the set of runs of m on σ. (This set has only one element if m is deterministic.) Define $INF_m(\sigma)$ to be the set of automaton states that appear infinitely often in any element of $\Gamma_m(\sigma)$. Then, σ is accepted by m if and only if $INF_m(\sigma) \cap Q_\infty \neq \emptyset$.

Examples of properties

A Buchi automaton m_{pc} that specifies *Partial Correctness* is shown in Fig. 4. As in m_{tc} (Fig. 1), *Pre* is a transition predicate that holds for states satisfying the given precondition, *Done* holds for states in which the program has terminated, and *Post* holds for states satisfying the given postcondition. Thus, m_{pc} accepts all sequences in which (i) the first state satisfies $\neg Pre$, as well as those (ii) consisting of a state satisfying *Pre*, followed by an infinite sequence of states satisfying $\neg Done$, and those (iii) consisting of a state satisfying *Pre*, followed by a (possibly empty) finite sequence of states sat-

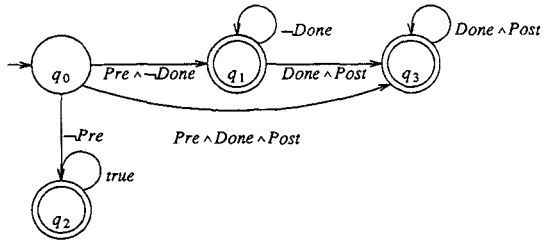

Fig. 4. m_{pc}

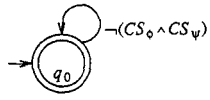

Fig. 5. m_{mutex}

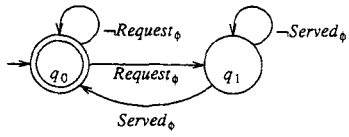

Fig. 6. m_{starv}

isfying $\neg Done$, followed by an infinite sequence of states satisfying $Done \wedge Post$.

A Buchi automaton m_{mutex} for *Mutual Exclusion* of two processes is given in Fig. 5. We assume transition predicate CS_ϕ (CS_ψ) holds for any state in which process ϕ (ψ) is executing in its critical section.

Starvation Freedom for a process using a mutual exclusion protocol is specified by m_{starv} of Fig. 6. Predicate $Request_\phi$ characterizes the state of a process ϕ whenever it attempts to enter its critical section, and ϕ makes progress when its state satisfies the predicate $Served_\phi$, which holds whenever ϕ enters its critical section.

3 Safety and liveness

Just as properties can be viewed in terms of proscribed "bad things" and prescribed "good things", so can Buchi automata. When a "bad thing" ("good thing") of the property occurs, we would expect a "bad thing" ("good thing") to happen in the recognizer for that property. The "bad thing" for a Buchi automaton is attempting an undefined transition, because if such a "bad thing" happens (in every run) while reading an input, the Buchi automaton will not accept that input. The "good thing" for a Buchi automaton is entering an accepting state infinitely often, because we require this "good thing" to happen for an input to be accepted. Having isolated these "bad things" and "good things", it is possible to give an automata-theoretic characterization of safety and liveness.

Since every Buchi automaton is equivalent to some reduced Buchi automaton, it suffices to consider only reduced Buchi automata.

Recognizing safety

To give an automata-theoretic characterization of safety properties, we require the following formal definition of safety from Alpern and Schneider (1985a). Consider a property P that stipulates that some "bad thing" does not happen. If a "bad thing" happens in an infinite sequence σ, then it must do so after some finite prefix and must be irremediable. Thus, if $\sigma \not\models P$, there is some prefix of σ (that includes the "bad thing") for which no extension to an infinite sequence will satisfy P. Taking the contrapositive of this, we get the formal definition of a safety property P:

Safety:

$$(\forall \sigma: \sigma \in S^\omega: \sigma \models P \Leftrightarrow (\forall i: 0 \leq i: (\exists \beta: \beta \in S^\omega: \sigma[..i]\,\beta \models P))), \quad (3.1)$$

where S is the set of program states, S^* the set of finite sequence of states, S^ω the set of infinite sequences of states, and juxtaposition is used to denote catenation of sequences.

For a reduced Buchi automaton m, define its *closure* $cl(m)$ to be the corresponding Buchi automaton in which every state has been made into an accepting state. For example, $Safe(m_{tc})$ (Fig. 7) is the closure of m_{tc} (Fig. 1) and m_{mutex} (Fig. 5) is its own closure. The closure of m can be used to determine whether the property specified by m is a safety property. This is because $cl(m)$ accepts a safety property – it never rejects an input by failing to enter accepting states (lack of a "good thing"); it rejects only by attempting an undefined transition (a "bad thing"). Thus, if a Buchi automaton m does not specify a safety property then $cl(m)$ accepts a proper superset of the inputs accepted by m. Therefore, if m and $cl(m)$ accept the same language then m recognizes a safety property.

Theorem 1. *A reduced Buchi automaton m specifies a safety property if and only if $L(m) = L(cl(m))$.*[2]

Proof. First, assume m specifies a safety property. Since $cl(m)$ is obtained from m by making all states accepting, every (infinite) sequence α accepted by m is also accepted by $cl(m)$. Hence, $L(m) \subseteq L(cl(m))$. To show $L(cl(m)) \subseteq L(m)$, we first prove that if $\alpha \in L(cl(m))$ then

$$(\forall i: 0 \leq i: (\exists \beta: \beta \in S^\omega: \alpha[..i]\,\beta \in L(m))). \quad (3.2)$$

Let $\alpha \in L(cl(m))$. Choose any prefix $\alpha[..i]$ and suppose $\alpha[..i]$ puts m in state q_i. Since m is reduced, q_i precedes an accepting state. Thus, there exists a sequence of program states β_0 that takes m from q_i to some accepting state q_{a1}. Moreover, there also exists a sequence of program states β_1 that takes m from q_{a1} to some accepting state q_{a2}, a sequence of program states β_2 that takes m from q_{a2} to some accepting state q_{a3}, etc. Define $\beta = \beta_0 \beta_1 \beta_2 \ldots$ Since $\alpha[..i]\beta$ causes m to be in accepting states infinitely often, $\alpha[..i]\ \beta \in L(m)$, so (3.2) is satisfied. Now, from (3.2) we can conclude

[2] See Sistla et al. 1985 for an algorithm to test whether the languages accepted by two Buchi automata are equal

$\alpha \in L(m)$ due to (3.1), because by assumption $L(m)$ is a safety property. Thus, $\alpha \in L(cl(m)) \Rightarrow \alpha \in L(m)$, hence $L(cl(m)) \subseteq L(m)$.

Next, assume $L(m) = L(cl(m))$. To show that m specifies a safety property, we show that Def. (3.1) is satisfied. Trivially,

$$\sigma \in L(m)$$
$$\Rightarrow (\forall i: 0 \leq i: (\exists \beta: \beta \in S^\omega: \sigma[..i]\ \beta \in L(m)))$$

since we can choose $\beta = \sigma[i+1..]$. Thus, it suffices to prove

$$\sigma \notin L(m)$$
$$\Rightarrow \neg (\forall i: 0 \leq i: (\exists \beta: \beta \in S^\omega: \sigma[..i]\ \beta \in L(m))). \quad (3.3)$$

Since $L(m) = L(cl(m))$, (3.3) is equivalent to

$$\sigma \notin L(cl(m))$$
$$\Rightarrow (\exists i: 0 \leq i: (\forall \beta: \beta \in S^\omega: \sigma[..i]\ \beta \notin L(cl(m)))).$$

Suppose $\sigma \notin L(cl(m))$. Thus, $cl(m)$ rejects σ. σ must be rejected by $cl(m)$ because an undefined transition is attempted since, by construction, every state in $cl(m)$ is accepting. Let this undefined transition occur upon reading $\sigma[k]$. Hence,

$$(\forall \beta: \beta \in S^\omega: \sigma[..k]\ \beta \notin L(cl(m)))$$

as required for m to specify a safety property according to Def. (3.1). □

For example, according to Theorem 1, m_{pc} (Fig. 4), m_{mutex} (Fig. 5), and $Safe(m_{tc})$ (Fig. 7) all specify safety properties.

Recognizing liveness

To give an automata-theoretic characterization of liveness properties, we require the following formal definition of liveness from Alpern and Schneider (1985a). The thing to observe about a liveness property is that no partical execution is irremediable since if some partial execution were irremediable, then it would be a "bad thing". We take this to be the defining characteristic of liveness. Thus, P is a liveness property if and only if

$$\textbf{Liveness:} (\forall \alpha: \alpha \in S^*: (\exists \beta: \beta \in S^\omega: \alpha\beta \models P)). \quad (3.4)$$

The closure of a Buchi automaton can also be used to determine if m specifies a liveness property. This is because $cl(m)$ can reject an input only by attempting an undefined transition – a "bad thing". A liveness property never proscribes a "bad thing", so if m specifies a liveness property, $cl(m)$ must accept every input.

Theorem 2. *A reduced Buchi automaton m specifies a liveness property if and only if $L(cl(m)) = S^\omega$.*

Proof. First, assume m specifies a liveness property. According to Def. (3.4) of liveness,

$$(\forall \alpha: \alpha \in S^\omega: (\forall i: 0 \leq i: (\exists \beta: \beta \in S^\omega: \alpha[..i]\ \beta \in L(m)))).$$

Thus, m does not attempt an undefined transition when reading an input α. Since $cl(m)$ has the same transition function as m, $cl(m)$ does not attempt an undefined transition when reading α. Each of the states of $cl(m)$ is accepting, and thus $cl(m)$ accepts α. Hence, $(\forall \alpha: a \in S^\omega: \alpha \in L(cl(m)))$, or equivalently $L(cl(m)) = S^\omega$, as required.

Next assume $L(cl(m)) = S^\omega$. Thus, $cl(m)$, hence m, does not attempt an undefined transition when reading any input α. Therefore, m does not attempt an undefined transition when reading a finite prefix $\alpha[..i]$ of an input. Suppose $\alpha[..i]$ leaves m in automation state q_i. Since m is reduced, there exists a path from q_i to some accepting state q_{a1}, from q_{a1} to some accepting state q_{a2}, etc. Let β_0 be a sequence of program states that takes m from q_i to q_{a1}, let β_1 be a sequence of program states that takes m from q_{a1} to q_{a2}, etc. Thus, $(\exists \beta: \beta \in S^\omega: \alpha[..i]\ \beta \in L(m))$ since we can choose $\beta = \beta_0 \beta_1 \ldots$ Definition (3.4) of liveness therefore holds, so $L(m)$ is a liveness property. □

For example, according to Theorem 2, m_{mono} (Fig. 2), m_{starv} (Fig. 6), and $Live(m_{tc})$ (Fig. 8) specify liveness properties.

4 Partitioning into safety and liveness

Given a Buchi automaton m, it is not difficult to construct Buchi automata $Safe(m)$ and $Live(m)$ such that $Safe(m)$ specifies a safety property, $Live(m)$ specifies a liveness property, and the property specified by m is the intersection of those specified by $Safe(m)$ and $Live(m)$. This proves that every property specified by a Buchi automaton is equivalent to the conjunction of a safety property and a liveness property, each of which can be specified by a Buchi automaton.

For $Safe(m)$, we use $cl(m)$.

Theorem 3. *$Safe(m)$ specifies a safety property.*

Proof. By construction, $cl(cl(m)) = cl(m)$. Hence, $L(cl(cl(m))) = L(cl(m))$, and the result follows from Theorem 1. □

We will construct $Live(m)$ to be a Buchi automaton such that

$$L(Live(m)) = L(m) \cup (S^\omega - L(cl(m))). \quad (4.1)$$

If m is deterministic then for $Live(m)$ we use m augmented by an accepting trap state[3] q_{trap}. Transition function $\delta_{Live(m)}$ is the transition function for m extended so that it causes every undefined transition of m to put $Live(m)$ in q_{trap}.

If m is non-deterministic then for $Live(m)$ we use use $m \times env(m)$ and take as accepting states any automation state in which m or $env(m)$ is in an accepting state. Buchi automaton $env(m)$ specifies $S^\omega - L(cl(m))$ and is defined as follows. The states of $env(m)$ include a trap state q_{trap} and the sets of automaton states of m; q_{trap} is the only accepting state of $env(m)$. The transition function $\delta_{env(m)}$ is, for all sets q_i of the automaton states of m:

$$\delta_{env(m)}(q_i, s) = \begin{cases} q_{trap} \text{ if } \delta_m(q_i, s) = \emptyset \\ \delta_m(q_i, s) \text{ otherwise} \end{cases}$$

Lemma. $L(Live(m)) = L(m) \cup (S^\omega - L(cl(m)))$.

Proof. First, consider the case where m is deterministic. Assume $\alpha \in L(Live(m))$. Thus, $Live(m)$ enters accepting states infinitely often when reading α. If all of these accepting states are also (accepting) states of m then m will accept α, hence $\alpha \in L(m)$. Otherwise, one of the accepting states is q_{trap}. In this case, $\alpha \notin L(cl(m))$ because to enter q_{trap}, m makes an undefined transition. From $\alpha \notin L(cl(m))$, we conclude $\alpha \in (S^\omega - L(cl(m)))$.

Now assume $\alpha \in L(m) \cup (S^\omega - L(cl(m)))$. If $\alpha \in L(m)$ then, by construction, $\alpha \in L(Live(m))$. If $\alpha \in (S^\omega - L(cl(m)))$ then $\alpha \notin L(cl(m))$, hence $cl(m)$ attempts an undefined transition on input α. This means that m attempts an undefined transition of input α, so, by construction, $Live(m)$ will make a transition to q_{trap} and accept α. Hence, $\alpha \in L(Live(m))$.

Next, consider the case where m is non-deterministic. By construction of the accepting states of $Live(m)$, $L(Live(m)) = L(m) \cup L(env(m))$. Since $env(m)$ specifies $S^\omega - L(cl(m))$, the result follows. □

We can now prove that $Live(m)$ specifies a liveness property. By construction, $Live(m)$ never rejects an input by attempting an undefined transition (a "bad thing"). Because m is reduced and $Live(m)$ accepts any input accepted by m, for each finite sequence of program states there is an infinite suffix that can be appended to obtain a sequence that will be accepted by $Live(m)$ – the defining characteristic of liveness.

[3] A *trap state* is one that has a transition to itself for all inputs

Theorem 4. *$Live(m)$ specifies a liveness property.*

Proof. First, consider the case where m is deterministic. By construction, $Live(m)$ has no undefined transitions. Therefore $L(cl(Live(m))) = S^\omega$ and the result follows from Theorem 2.

Next, consider the case where m is non-deterministic. Due to the construction of $Live(m)$,

$$L(cl(Live(m))) = L(cl(m)) \cup L(cl(env(m))).$$

$L(cl(env(m))) \supseteq L(env(m))$, since $cl(env(m))$ differs from $env(m)$ by having more states accepting. Substituting,

$$L(cl(Live(m))) \supseteq L(cl(m)) \cup L(env(m)).$$

Since, by construction, $L(env(m)) = S^\omega - L(cl(m))$, we get

$$L(cl(Live(m))) \supseteq L(cl(m)) \cup (S^\omega - L(cl(m))),$$

or $L(cl(Live(m))) \supseteq S^\omega$. Since $L(cl(Live(m))) \subseteq S^\omega$, the result then follows from Theorem 2. □

Finally, we can prove

Theorem 5. *Given a reduced Buchi automaton m, $L(m) = L(Safe(m)) \cap L(Live(m))$.*

Proof. Substituting for $L(Live(m))$ according to (4.1) and for $L(Safe(m))$ according to Theorem 1, we get

$$\begin{aligned} &L(Live(m)) \cap L(Safe(m)) \\ &= (L(m) \cup (S^\omega - L(cl(m))) \cap L(cl(m)) \\ &= (L(m) \cap L(cl(m))) \cup ((S^\omega - L(cl(m))) \cap L(cl(m))) \\ &= (L(m) \cap L(cl(m))) \cup \emptyset \\ &= L(m) \quad \square \end{aligned}$$

The construction of Theorem 5 is illustrated below for m_{tc} of Fig. 1, which specifies Total Correctness. $Safe(m_{tc})$ is given in Fig. 7; $Live(m_{tc})$ is given in Fig. 8. However, $Live(m_{tc})$ can be simplified by combining the three accepting states q_2, q_3, and q_t into a single accepting state. Combining these states results in an equivalent automaton because once any of these accepting states is entered,

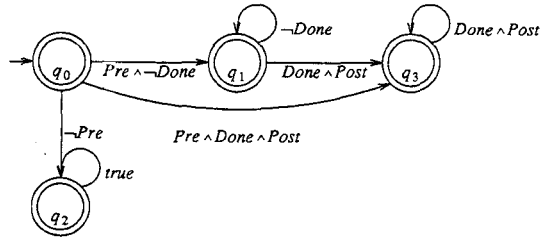

Fig. 7. $Safe(m_{tc})$

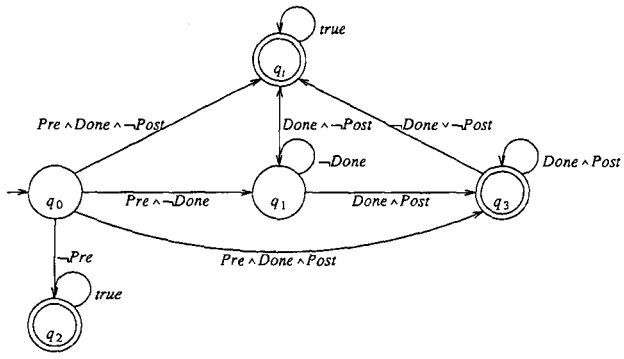

Fig. 8. $Live(m_{tc})$

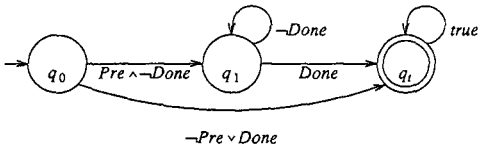

Fig. 9. $Live\ (m_{tc})$ simplified

$Live(m_{tc})$ will thereafter remain in accepting states. Once q_2 or q_1 is entered, $Live(m_{tc})$ will remain in that state since each of those states has a transition to itself for all inputs. And, if accepting state q_3 is entered then either $Live(m_{tc})$ will forever remain in q_3 because subsequent input symbols satisfy $Done \wedge Post$ or some input symbol satisfying $\neg (Done \wedge Post)$ must have been read, causing a transition to (accepting state) q_t, which has a transition to itself for all inputs. The (equivalent) automaton that results from combining the three accepting states of $Live(m_{tc})$ is given in Fig. 9.

Notice that $L(Safe(m_{tc})) = L(m_{pc})$ because $Safe(m_{tc})$ differs from m_{pc} only by having q_0 as an accepting state. Thus, we have shown that Partial Correctness (as specified by m_{pc} of Fig. 4) is the safety component of Total Correctness. Also, observe that the simplified $Live(m_{tc})$ automaton (Fig. 9) specifies a property comprising sequences of program states in which either the first state satisfies $\neg Pre$ or eventually there is a state satisfying *Done*. Such a property might well be called *Termination*, since it requires the program to reach *Done* if started in a state satisfying *Pre*. Termination is the liveness component of Total Correctness. We have therefore proved, by our construction, that Total Correctness is the intersection of Partial Correctness and Termination.

5 Proving deterministic safety and liveness properties

Given a deterministic Buchi automaton specification of a property, it is possible to extract proof obligations that must be satisfied by any program for which that property holds. This forms the basis for an approach to program verification first proposed in Alpern and Schneider (1985a) and Alpern (1986)[4], and permits us to formalize the relationship of safety and liveness properties specified by deterministic Buchi automata to invariance and well-foundedness arguments.

Let m be a deterministic property recognizer for property P. One can think of m as simulating – in an abstract way – any program that satisfies P. To show that a program π satisfies m, we demonstrate a correspondence between m and π. We do this by defining a *correspondence invariant* C_i for each automaton state q_i, where C_i is a predicate that holds on a program state s if there exists a history of π containing a program state s and m enters q_i upon reading s. Thus, if m is ever in automaton state q_i, the last program state it read must satisfy C_i. Constraints satisfied by correspondence invariants are defined inductively, as follows.

For the base case, initially, m is in state q_0 and π is in a state characterized by $Init_\pi$. Suppose that upon reading s_0, the first program state of some history of π, m enters automaton state q_j. Thus, s_0 satisfies $Init_\pi$ and T_{0j}, the transition predicate labeling the edge that connects q_0 and q_j. Therefore, C_j must satisfy $(Init_\pi \wedge T_{0j}) \Rightarrow C_j$; for any automaton state q_j entered upon reading the first symbol of any history of π, we require

Correspondence Basis:

$$(\forall j: q_j \in Q: (Init_\pi \wedge T_{0j}) \Rightarrow C_j). \tag{5.1}$$

Next we must prove the induction step. Assume that if m enters automaton state q_i upon reading program state s_k in a history of π and $0 \leq k < K$, then s_k satisfies C_i. Consider the case when m reads s_K. Suppose m is in state q_i and that upon reading program state s_K, a transition is made to automaton state q_j. By the induction hypothesis s_{K-1} satisfied C_i and s_K satisfies transition predicate T_{ij}. The appropriate correspondence invariant C_j will hold provided $\{C_i\}\ \alpha \{T_{ij} \Rightarrow C_j\}$ is valid for any α, an atomic action of π. (If α is not enabled in s_{K-1} then the triple is trivially valid because an atomic action terminates only when started in a state in which it is enabled.) Generalizing to handle any atomic action and any automaton state that m might be in when s_K is read, we require:

[4] The approach has since been extended to handle properties specified by non-deterministic automata (Manna and Pnueli 1987; Alpern and Schneider 1987)

Correspondence Induction:
For all α: $\alpha \in \mathscr{A}_\pi$:
For all i: $q_i \in Q$:

$$\{C_i\}\,\alpha\,\{\bigwedge_{j:\,q_j \in Q} (T_{ij} \Rightarrow C_j)\}. \tag{5.2}$$

Thus, any collection of predicates satisfying (5.1) and (5.2) are correspondence invariants for m and π.

In order to establish that π satisfies P, we must show that every history of π is accepted by m. There are two ways that m might fail to accept a history σ or π:

(1) m attempts an undefined transition when reading σ.

(2) m never enters an accepting state after some finite prefix of σ.

Thus, in order to prove that every history of π satisfies P, it suffices to show that (1) and (2) are impossible.

Two obligations ensure that (1) is impossible. First, we must show that m can make some transition from its start state upon reading the first program state in a history:

Transition Basis: $$Init_\pi \Rightarrow \bigvee_{j:\,q_j \in Q} T_{0j}. \tag{5.3}$$

Second, we must show that m can always make a transition upon reading subsequent states in a history. If m is in state q_i then the program state just read by m satisfies a correspondence invariant C_i. To avoid an undefined transition, any atomic action α that is then executed must transform the program state so that one of the transition predicates T_{ij} emanating from q_i holds. This is guaranteed by

Transition Induction: For all α: $\alpha \in \mathscr{A}_\pi$:
For all i: $q_i \in Q$:

$$\{C_i\}\,\alpha\,\{\bigvee_{j:\,q_j \in Q} T_{ij}\}. \tag{5.4}$$

Finally, we derive conditions to ensure that m cannot reject σ by failing to enter accepting states infinitely often (i.e., scenario (2) above). A set Q' of automaton states is *strongly connected* if and only if there is a sequence of transitions from any element of Q' to any other without involving an automaton state outside of Q'. A *reject knot* κ is a maximal strongly connected subset of Q containing no accepting states. To ensure that m does not reject σ, we must prove that no run for a history of π is restricted to automaton states in $Q - Q_{inf}$. We do this by constructing a *variant function* v_κ for each reject knot κ.

A variant function $v_\kappa(q, s)$ is a function from automaton and program states to some well-founded set.[5] For simplicity, assume that this well-founded set is the Natural Numbers. We require that whenever $v_\kappa(q, s) = 0$ for any automata state q and program state s, then q is not in κ.

Knot Exit: $$(\forall i: q_i \in \kappa: (v_\kappa(q_i) = 0) \Rightarrow \neg C_i). \tag{5.5}$$

And, to ensure that the variant function does reach 0, we require that it be decreased by every atomic action in π that might be executed:

Knot Variance:
For all α: $\alpha \in \mathscr{A}_\pi$:
For all $q_i \in \kappa$:

$$\{C_i \wedge 0 < v_\kappa(q_i) = \mathrm{V}\}\quad \alpha\quad \{\bigwedge_{j:\,q_j \in \kappa} ((T_{ij} \wedge C_j) \Rightarrow v_\kappa(q_j) < \mathrm{V})\}. \tag{5.6}$$

The six proof obligations – Correspondence Basis (5.1), Correspondence Induction (5.2), Transition Basis (5.3), Transition Induction (5.4), Knot Exit (5.5), and Knot Variance (5.6) – are of three basic forms. Correspondence Basis (5.1), Transition Basis (5.3), and Knot Exit (5.5) involve proving that predicate logic formulas are valid. Correspondence Induction (5.2) and Transition Induction (5.4) involve proving invariance of assertions. And, Knot Variance (5.6) involves proving that certain events cause variant functions to be decreased. There is a relationship between safety and liveness properties and these three forms of proof obligations.

First, observe that by construction, *Safe*(m) cannot have reject knots. Thus, Knot Exit (5.5) and Knot Variance (5.6) are trivially satisfied, so proving that a program satisfies *Safe*(m) never requires a variant function (or well-foundedness argument). The remaining proof obligations for *Safe*(m) constitute an invariance argument. Moreover, if m specifies a safety property then $L(m) = L(Safe(m))$. Proving than π satisfies *Safe*(m) is, therefore, sufficient in order to prove that π satisfies m. Thus, safety properties specified by deterministic Buchi automata can be proved using only invariance arguments.

Second, recall from Theorem 2 that m specifies a liveness property if $L(cl(m)) = S^\omega$. By construction, if $L(cl(m)) = S^\omega$ then it is not possible for m to attempt an undefined transition. Therefore, Transition Basis (5.3) and Transition Induction

[5] The program state argument is often left implicit

(5.4) are trivially satisfied when trying to prove that a program π satisfies a liveness property. Automata specifying liveness properties can have reject knots, so Knot Exit (5.5) and Knot Variance (5.6) must be proved – a variant function of well-foundedness argument is therefore required in proving a liveness property. In addition, an invariance argument is required to satisfy (5.1) and (5.2).

6 Related work

The first formal definition of safety was given in Lamport (1985). While that definition correctly captures the intuition for an important class of safety properties – those invariant under stuttering – it is inadequate for safety properties that are not invariant under stuttering. The formal definition of safety used in this paper, which was first proposed in Alpern and Schneider (1985a), is independent of stuttering; in Alpern et al. (1985) it is shown equivalent to Lamport's for properties that are invariant under stuttering. The definition of liveness used in this paper also first appeared in Alpern and Schneider (1985a) along with a proof that every property can be expressed as the conjunction of a safety property and a liveness property. That proof is based on a topological characterization of safety properties as closed sets and liveness properties as dense sets. The automata-theoretic characterizations and proofs in this paper more closely parallel the informal definitions of safety and liveness in terms of "bad things" and "good things".

In Sistla (1985), an attempt is made to characterize syntactically safety and liveness properties that are expressed in linear-time temporal logic. Deductive systems are given for safety and liveness formulas in a temporal logic with *eventually* ($\Diamond$), but without *next* ($\bigcirc$), or *until*. In addition, deductive systems for full (propositional) temporal logic are given for a subset of the safety properties, called strong safety properties, and for a subset of the liveness propertics, called absolute liveness properties. Finally, Sistla (1985) characterizes automata for safety properties as those whose states can be partitioned as "good" and "bad" such that no "bad" state is ever entered on an accepted input.[6] Sistla independently developed a method similar to ours for determining whether an automaton specifies a liveness property Sistla (1986). In Sistla's method, the closure of the reduced automaton is treated as a automaton on finite strings. If this automaton accepts S^* the original automaton specifies a liveness property. Also Sistla (1986) gives a syntactic characterization of the temporal logic specification of strong safety properties that can be defined using Buchi automata.

Another syntactic characterization of safety and liveness properties appears in Lichtenstein et al. (1985). The definition of safety given there coincides with ours; the definition of liveness classifies some properties as liveness that our definition does not. We do not classify *p until q* as liveness because the occurrence of $\neg p$ before q constitutes a "bad thing" and therefore *p until q* has elements of safety, but Lichtenstein et al. (1985) consider it liveness. Another difference is that the definitions in Lichtenstein et al. (1985) are based on existing temporal logic inference rules (proof obligations) whereas our definitions are not. Independence from inference rules makes our results about the relationship between types of properties and proof techniques all the more interesting. Also, in contrast to the definitions in Lichtenstein et al. (1985), our characterizations of safety and liveness are independent of the notation used to express the properties and therefore apply to a large class of properties.

7 Conclusions

This paper gives tests to determine whether a property specified by a (deterministic or nondeterministic) Buchi automaton is safety or liveness. For reduced deterministic Buchi automata, these tests are quite simple: such an automaton accepts a safety property if and only if each state is accepting (or could be made accepting without increasing the set of sequences accepted); such an automaton accepts a liveness property if and only if it has a transition defined from each state for each input symbol. We also show how to extract automata *Safe*(m) and *Live*(m) from a Buchi automaton m, where *Safe*(m) specifies a safety property, *Live*(m) specifies a liveness property, and the property specified by m is the intersection of the ones specified by *Safe*(m) and *Live*(m). The extraction is illustrated by proving that Total Correctness is the conjunction of safety property Partial Correctness and liveness property Termination. Finally, we prove that for properties specified by deterministic Buchi automata, safety properties can be proved by use of an invariance argument while liveness properties also require a well-foundedness argument.

[6] Sistla proves this result for deterministic Muller automata. In general, such automata are more powerful than deterministic Buchi automata. However, a simple consequence of this automata characterization of safety properties is that any deterministic Muller automata for a safety property is isomorphic to an equivalent deterministic Buchi automaton and vice versa

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