

Introduction

Modelling parallel systems

Linear Time Properties

Regular Properties

Linear Temporal Logic (LTL)

 syntax and semantics of LTL

 automata-based LTL model checking ←

 complexity of LTL model checking

Computation-Tree Logic

Equivalences and Abstraction

given: finite transition system \mathcal{T} over AP
(without terminal states)
LTL-formula φ over AP

question: does $\mathcal{T} \models \varphi$ hold ?

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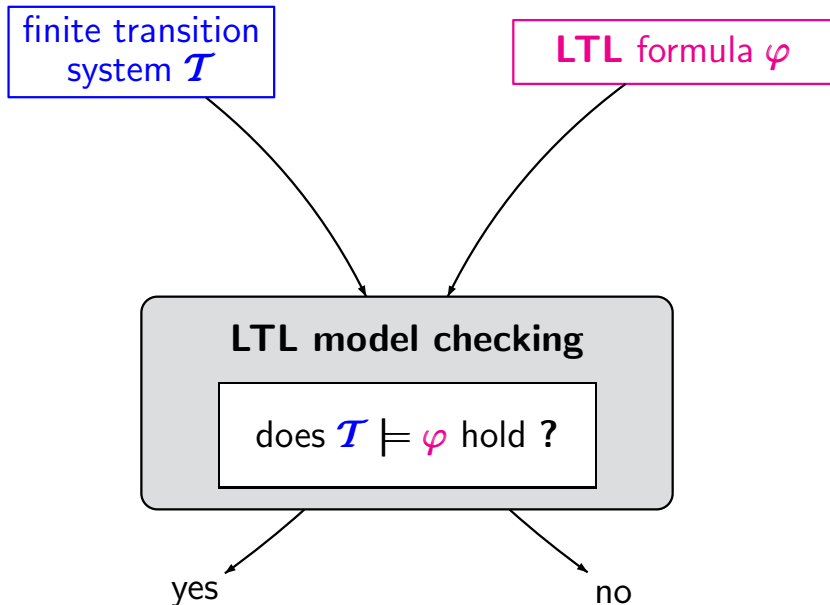
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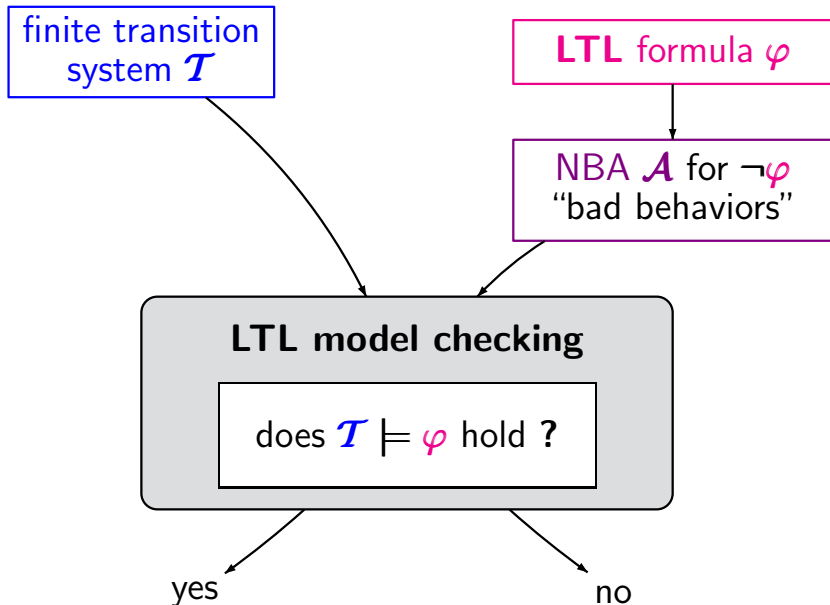
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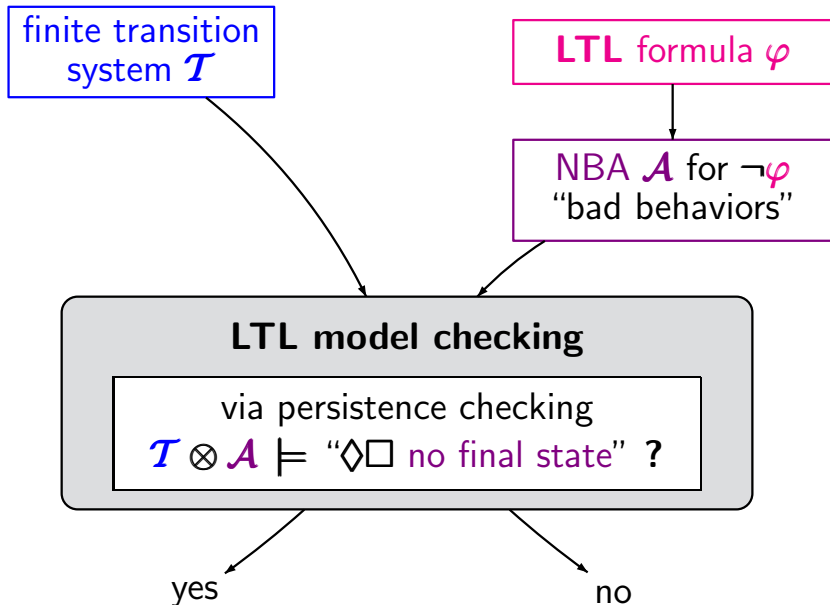
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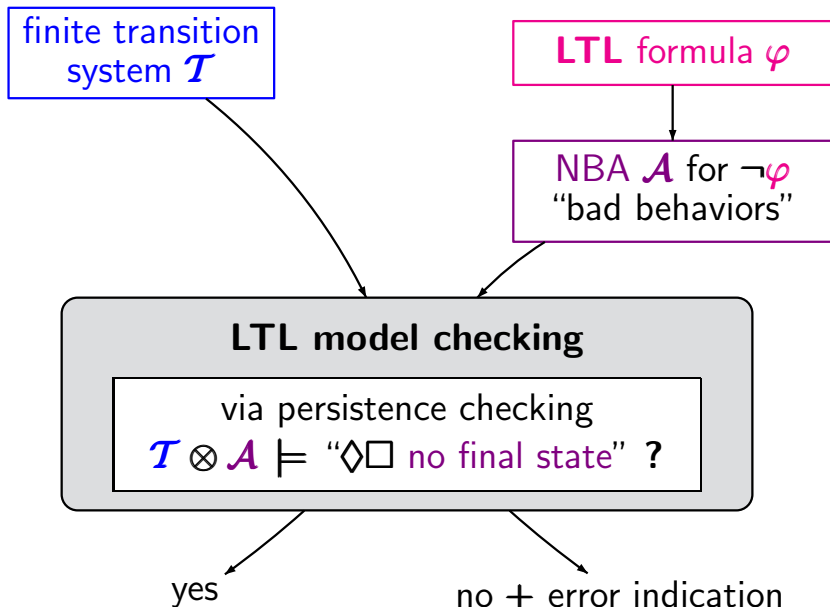


construct the product-TS $\mathcal{T} \otimes \mathcal{A}$
search a path in the product that meets
the acceptance condition of \mathcal{A}









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prefix of a path π

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$\mathcal{T} \models$ safety property E

iff $\text{Traces}_{fin}(\mathcal{T}) \cap \mathcal{L}(\mathcal{A}) = \emptyset$

where \mathcal{A} is an NFA for the bad prefixes

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iff $\mathcal{T} \otimes \mathcal{A} \models \square \neg F \leftarrow$ invariant checking

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iff $\mathcal{T} \otimes \mathcal{A} \models \diamond \square \neg F \leftarrow$ persistence checking

NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

- Q finite set of states
- Σ alphabet
- $\delta : Q \times \Sigma \rightarrow 2^Q$ transition relation
- $Q_0 \subseteq Q$ set of initial states
- $F \subseteq Q$ set of **final states**, also called **accept states**

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run for a word $A_0 A_1 A_2 \dots \in \Sigma^\omega$:

state sequence $\pi = q_0 q_1 q_2 \dots$ where $q_0 \in Q_0$
and $q_{i+1} \in \delta(q_i, A_i)$ for $i \geq 0$

run π is **accepting** if $\exists i \in \mathbb{N}. q_i \in F$

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accepted language $\mathcal{L}_\omega(\mathcal{A}) \subseteq \Sigma^\omega$ is given by:

$\mathcal{L}_\omega(\mathcal{A}) \stackrel{\text{def}}{=} \text{set of infinite words over } \Sigma \text{ that have an accepting run in } \mathcal{A}$

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$$\text{Words}(\varphi) = \mathcal{L}_\omega(\mathcal{A})$$

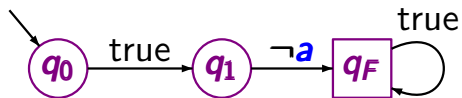
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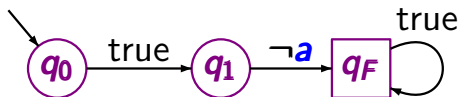
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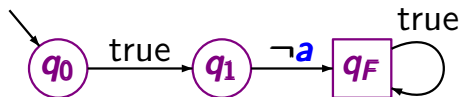
proof: ... later ...



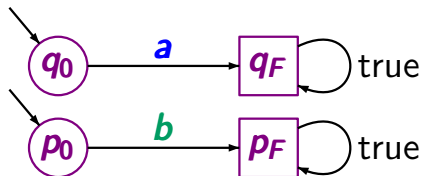
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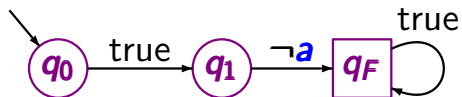
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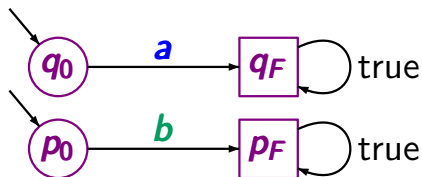
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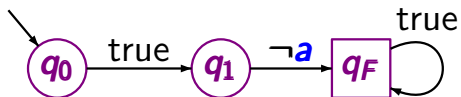
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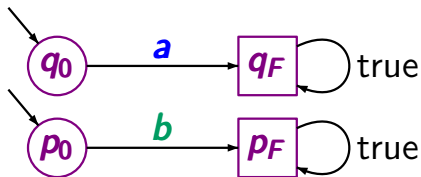
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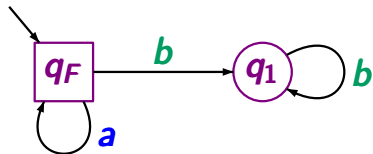
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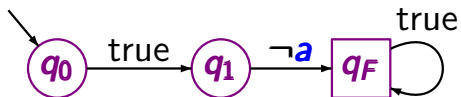
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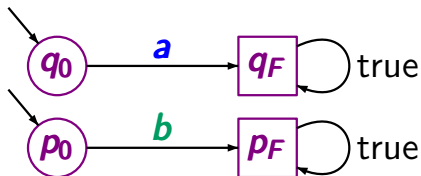
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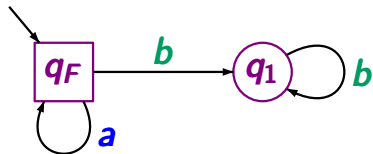
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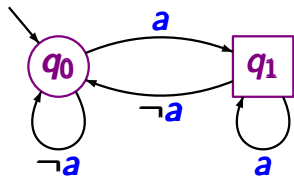
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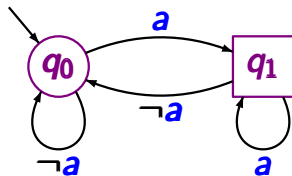
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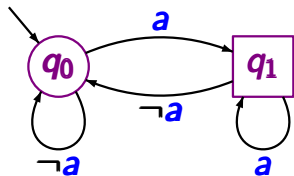
$$\mathcal{L}_\omega(\mathcal{A}) = \text{Words}(\Box a)$$



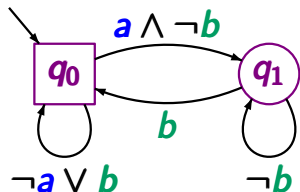
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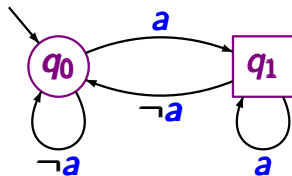
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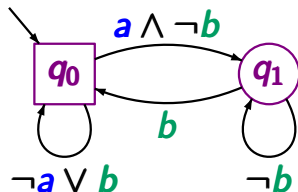
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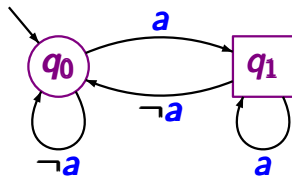


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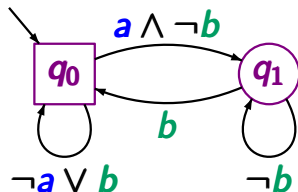


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e.g., $\emptyset\emptyset\emptyset\emptyset\dots = \emptyset^\omega$ } are accepted by \mathcal{A}
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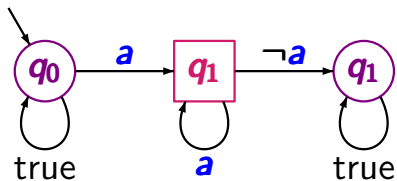


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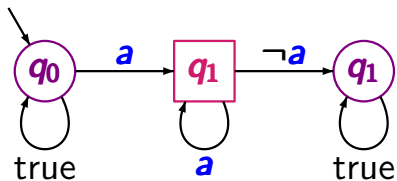


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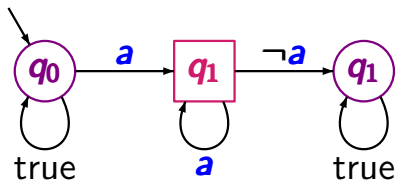
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possible runs for $\{a\}^\omega$

$q_0 \ q_0 \ q_0 \ q_0 \ q_0 \ q_0 \ \dots$

not accepting

$q_0 \ q_1 \ q_1 \ q_1 \ q_1 \ q_1 \ \dots$

accepting

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accepting

\vdots

Let A be an **NFA** for the language of all **bad prefixes** for a safety property E .

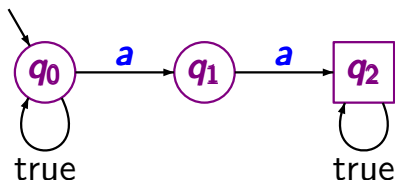
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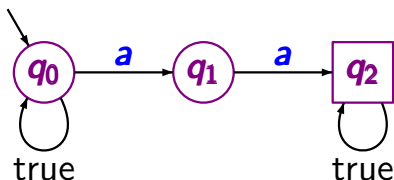
Example: $E \hat{=} \text{“never } a \text{ twice in a row”}$



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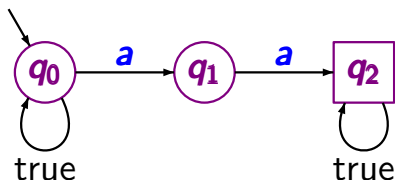
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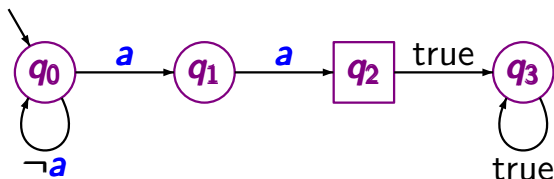
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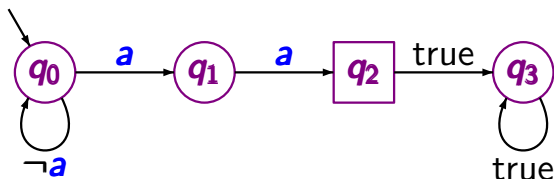
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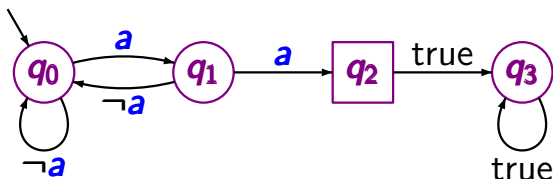
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Let \mathcal{A} be an **NFA** for the language of all bad prefixes for a safety property E . Then:

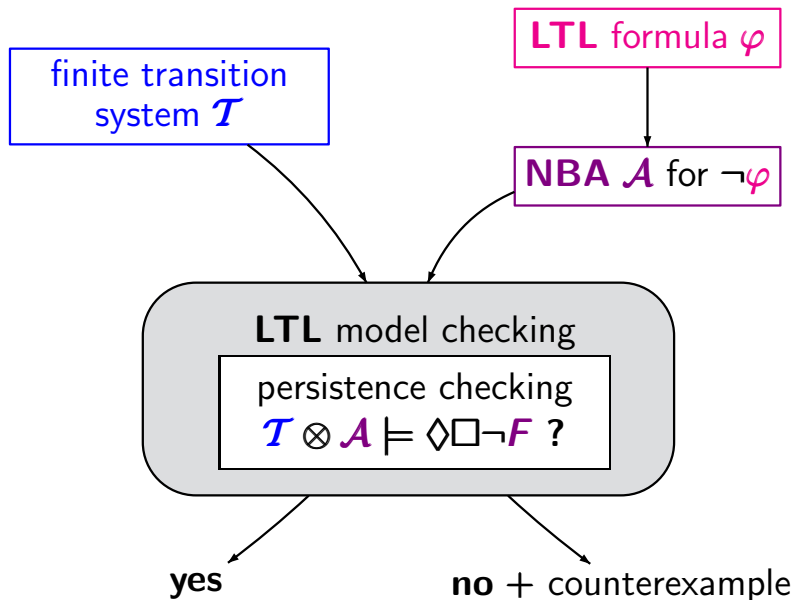
$$\mathcal{L}_\omega(\mathcal{A}) = \bar{E} = (2^{AP})^\omega \setminus E = \text{Words}(\neg\varphi)$$

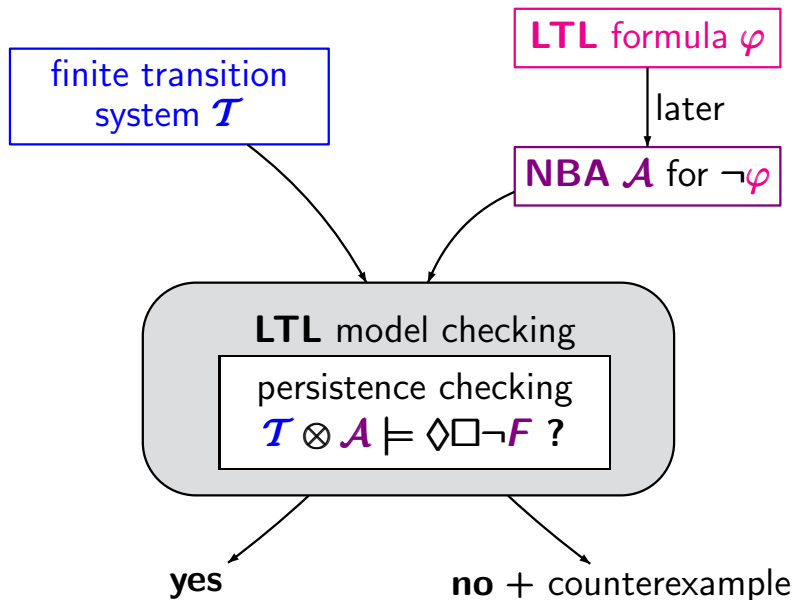
wrong, if $\mathcal{L}(\mathcal{A}) =$ language of minimal bad prefixes even if \mathcal{A} is a non-blocking DFA

Example: $E \hat{=} \text{“never } a \text{ twice in a row”}$



$$\mathcal{L}_\omega(\mathcal{A}) = \emptyset$$





$\mathcal{T} = (\mathcal{S}, Act, \rightarrow, S_0, AP, L)$ TS without terminal states

$\mathcal{A} = (\mathcal{Q}, 2^{AP}, \delta, Q_0, F)$ NBA or NFA

non-blocking, $Q_0 \cap F = \emptyset$

$\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ TS without terminal states

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product-TS $\mathcal{T} \otimes \mathcal{A} \stackrel{\text{def}}{=} (S \times Q, Act, \rightarrow', S'_0, AP', L')$

$\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ TS without terminal states

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product-TS $\mathcal{T} \otimes \mathcal{A} \stackrel{\text{def}}{=} (S \times Q, Act, \rightarrow', S'_0, AP', L')$

initial states: $S'_0 = \{\langle s_0, q \rangle : s_0 \in S_0, q \in \delta(Q_0, L(s_0))\}$

labeling: $AP' = Q, L'(\langle s, q \rangle) = \{q\}$

Recall: product transition system

LTLMC3.2-7

$\mathcal{T} = (\mathcal{S}, Act, \rightarrow, \mathcal{S}_0, AP, L)$ TS without terminal states

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non-blocking, $\mathcal{Q}_0 \cap F = \emptyset$

product-TS $\mathcal{T} \otimes \mathcal{A} \stackrel{\text{def}}{=} (\mathcal{S} \times \mathcal{Q}, Act, \rightarrow', \mathcal{S}'_0, AP', L')$

initial states: $\mathcal{S}'_0 = \{\langle s_0, q \rangle : s_0 \in \mathcal{S}_0, q \in \delta(\mathcal{Q}_0, L(s_0))\}$

labeling: $AP' = \mathcal{Q}, L'(\langle s, q \rangle) = \{q\}$

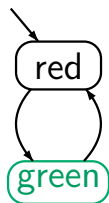
transition relation:

$$\frac{s \xrightarrow{\alpha} s' \wedge q' \in \delta(q, L(s'))}{\langle s, q \rangle \xrightarrow{\alpha'} \langle s', q' \rangle}$$

Example: LTL model checking

LTLMC3.2-8

TS \mathcal{T}

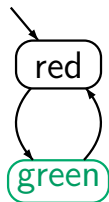


LTL formula $\varphi = \square \blacklozenge \text{green}$

Example: LTL model checking

LTLMC3.2-8

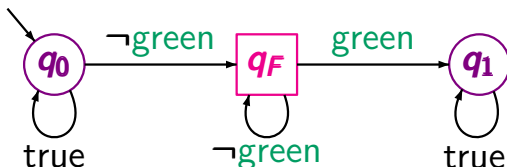
TS \mathcal{T}



LTL formula $\varphi = \Box\Diamond\text{green}$

NBA \mathcal{A} for the complement

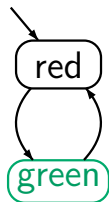
$\neg\varphi \equiv \Diamond\Box\neg\text{green}$



Example: LTL model checking

LTLMC3.2-8

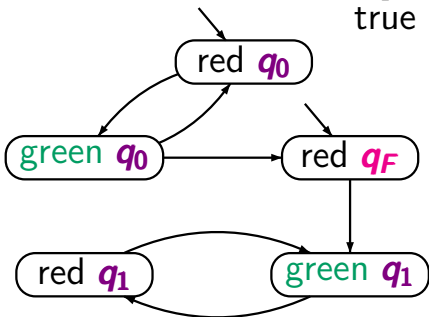
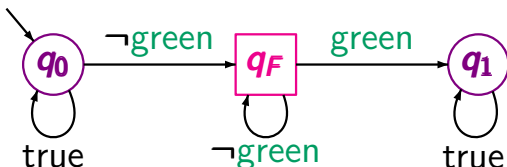
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LTL formula $\varphi = \Box\Diamond\text{green}$

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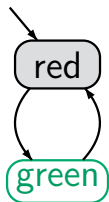


reachable fragment of the product TS $\mathcal{T} \otimes \mathcal{A}$

Example: LTL model checking

LTLMC3.2-8

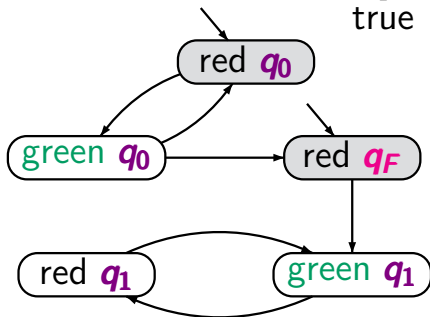
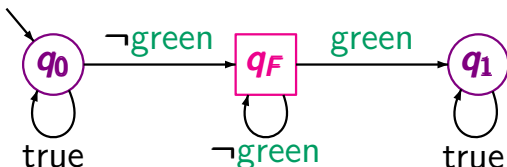
TS \mathcal{T}



LTL formula $\varphi = \Box\Diamond\text{green}$

NBA \mathcal{A} for the complement

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initial states:

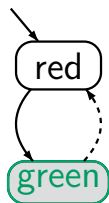
$\langle \text{red}, q \rangle$ where

$$\begin{aligned} q &\in \delta(q_0, L(\text{red})) \\ &= \delta(q_0, \emptyset) \\ &= \{q_0, q_F\} \end{aligned}$$

Example: LTL model checking

LTLMC3.2-8

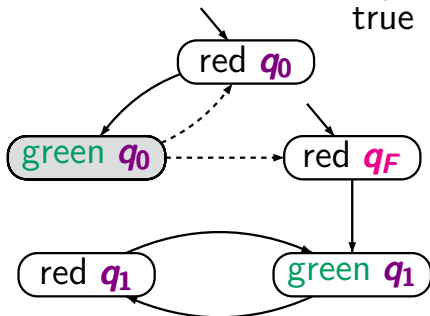
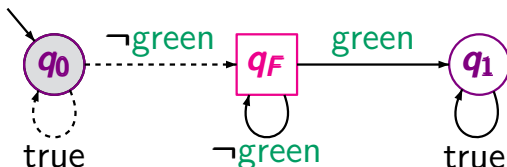
TS \mathcal{T}



LTL formula $\varphi = \Box\Diamond\text{green}$

NBA \mathcal{A} for the complement

$$\neg\varphi \equiv \Diamond\Box\neg\text{green}$$



transition

$$\langle \text{green}, q_0 \rangle \rightarrow \langle \text{red}, q \rangle$$

$$q \in \delta(q_0, L(\text{red}))$$

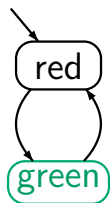
$$= \delta(q_0, \emptyset)$$

$$= \{q_0, q_F\}$$

Example: LTL model checking

LTLMC3.2-8

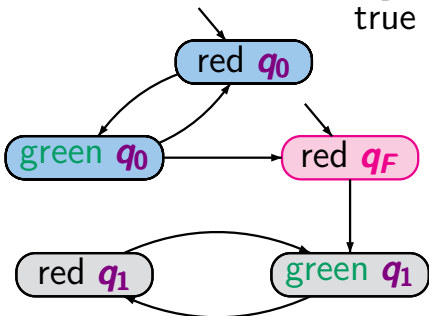
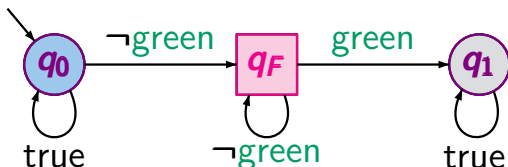
TS \mathcal{T}



LTL formula $\varphi = \Box\Diamond\text{green}$

NBA \mathcal{A} for the complement

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atomic propositions

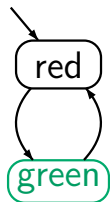
$$AP' = \{q_0, q_F, q_1\}$$

obvious labeling function

Example: LTL model checking

LTLMC3.2-8

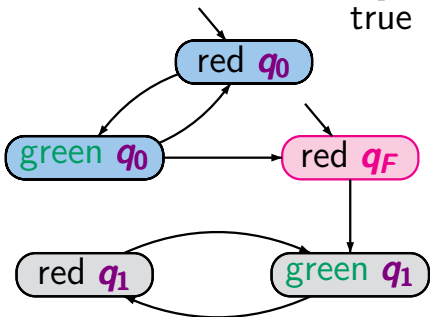
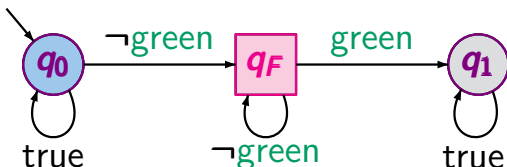
TS \mathcal{T}



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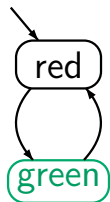


$\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$

Example: LTL model checking

LTLMC3.2-8

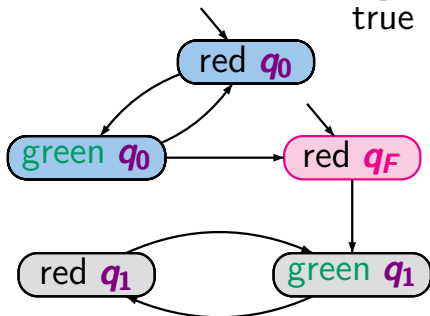
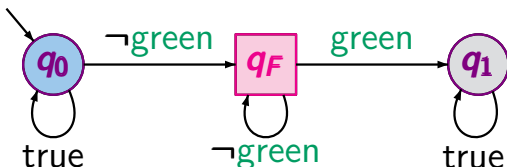
TS \mathcal{T}



LTL formula $\varphi = \Box\Diamond\text{green}$

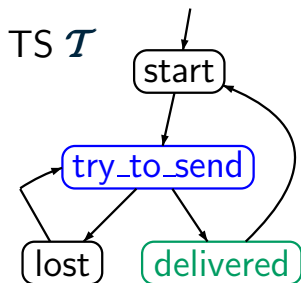
NBA \mathcal{A} for the complement

$$\neg\varphi \equiv \Diamond\Box\neg\text{green}$$



$$\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$$

$$\text{hence: } \mathcal{T} \models \varphi$$

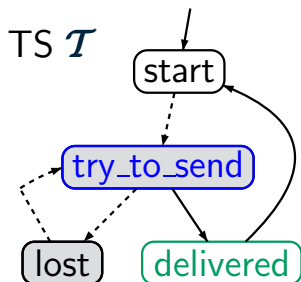


LTL formula $\varphi = \square(\text{try} \rightarrow \diamond \text{del})$

“each (repeatedly) sent message will eventually be delivered”

Example: LTL model checking

LTLMC3.2-9



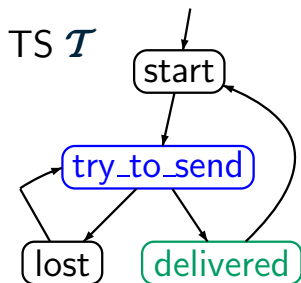
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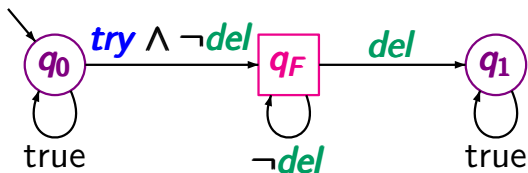
$\mathcal{T} \not\models \varphi$

Example: LTL model checking

LTLMC3.2-9



NBA \mathcal{A} for $\neg\varphi \equiv \diamond(\text{try} \wedge \square\neg\text{del})$



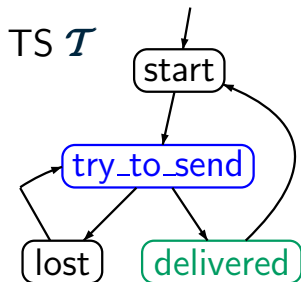
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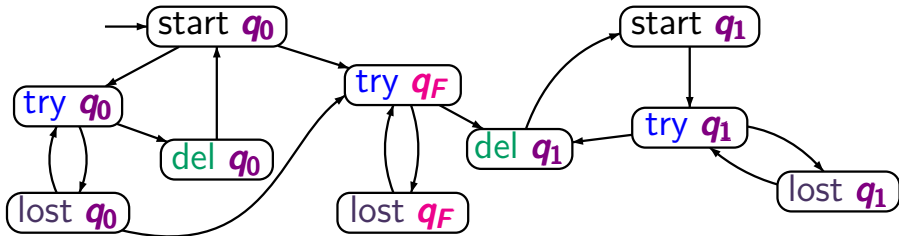
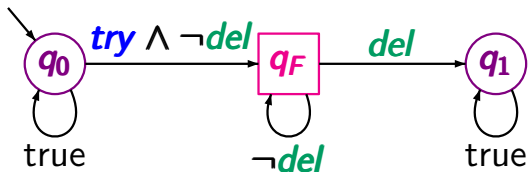
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Example: LTL model checking

LTLMC3.2-9



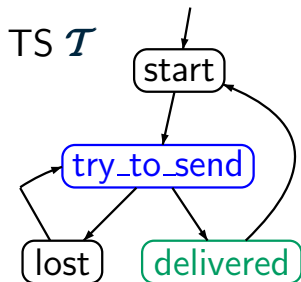
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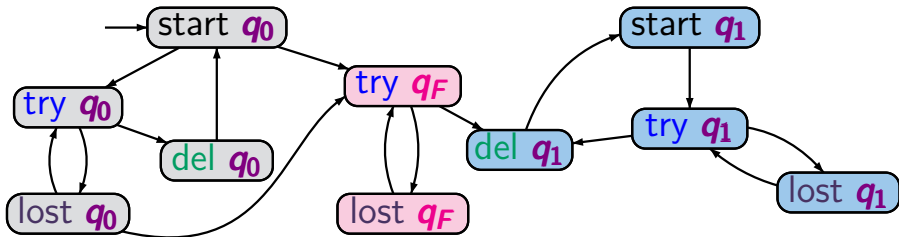
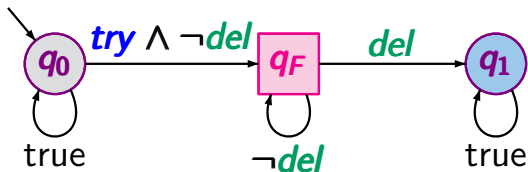
reachable fragment of the product-TS

Example: LTL model checking

LTLMC3.2-9



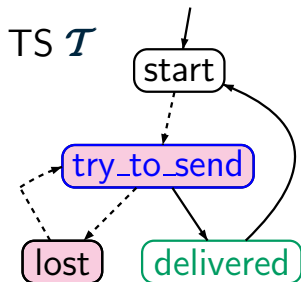
NBA \mathcal{A} for $\neg\varphi \equiv \diamond(\text{try} \wedge \square\neg\text{del})$



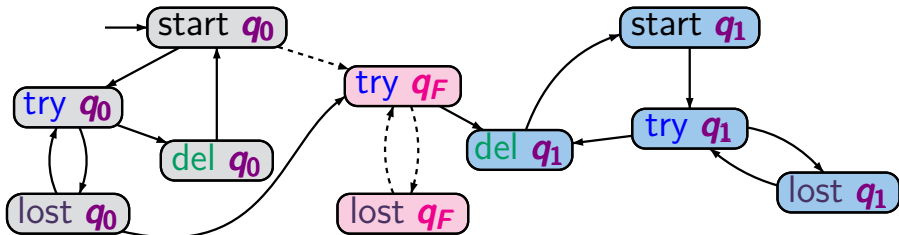
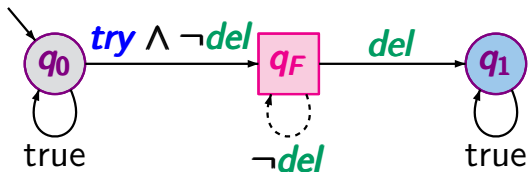
set of atomic propositions $AP' = \{q_0, q_1, q_F\}$

Example: LTL model checking

LTLMC3.2-9



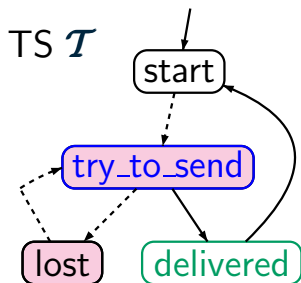
NBA \mathcal{A} for $\neg\varphi \equiv \Diamond(\text{try} \wedge \Box\neg\text{del})$



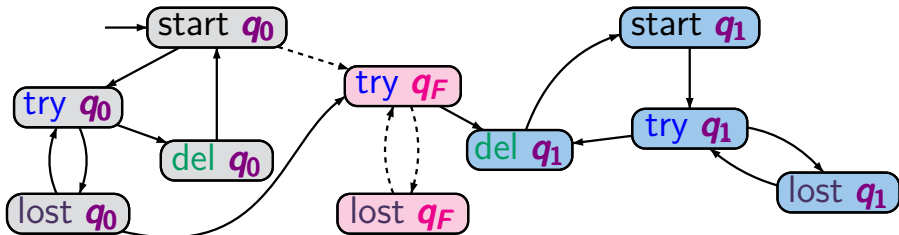
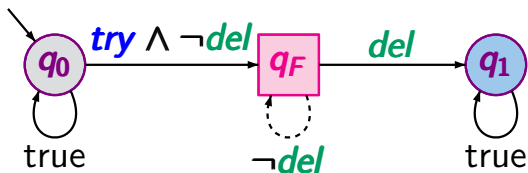
$$\mathcal{T} \otimes \mathcal{A} \not\models \Diamond \Box \neg F$$

Example: LTL model checking

LTLMC3.2-9



NBA \mathcal{A} for $\neg\varphi \equiv \diamond(\text{try} \wedge \square\neg\text{del})$



$\mathcal{T} \otimes \mathcal{A} \not\models \diamond\square\neg F$

hence: $\mathcal{T} \not\models \varphi$

given: finite TS \mathcal{T} , LTL-formula φ

question: does $\mathcal{T} \models \varphi$ hold ?

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construct an NBA \mathcal{A} for $\neg\varphi$ and the product $\mathcal{T} \otimes \mathcal{A}$

check whether $\mathcal{T} \otimes \mathcal{A} \models \diamond\Box\neg F$

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persistence
checking
nested **DFS**

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persistence
checking
nested **DFS**

IF $\mathcal{T} \otimes \mathcal{A} \models \diamond\Box\neg F$

THEN return “yes”

ELSE compute a counterexample

$\langle s_0, p_0 \rangle \dots \langle s_n, p_n \rangle \dots \langle s_n, p_n \rangle$

for $\mathcal{T} \otimes \mathcal{A}$ and $\diamond\Box\neg F$

return “no” and $s_0 \dots s_n \dots s_n$

given: finite TS \mathcal{T} , LTL-formula φ

question: does $\mathcal{T} \models \varphi$ hold ?

~~construct an NBA \mathcal{A} for $\neg\varphi$ and the product $\mathcal{T} \otimes \mathcal{A}$~~

check whether $\mathcal{T} \otimes \mathcal{A} \models \diamond\Box\neg F$ ← persistence checking nested **DFS**

IF $\mathcal{T} \otimes \mathcal{A} \models \diamond\Box\neg F$

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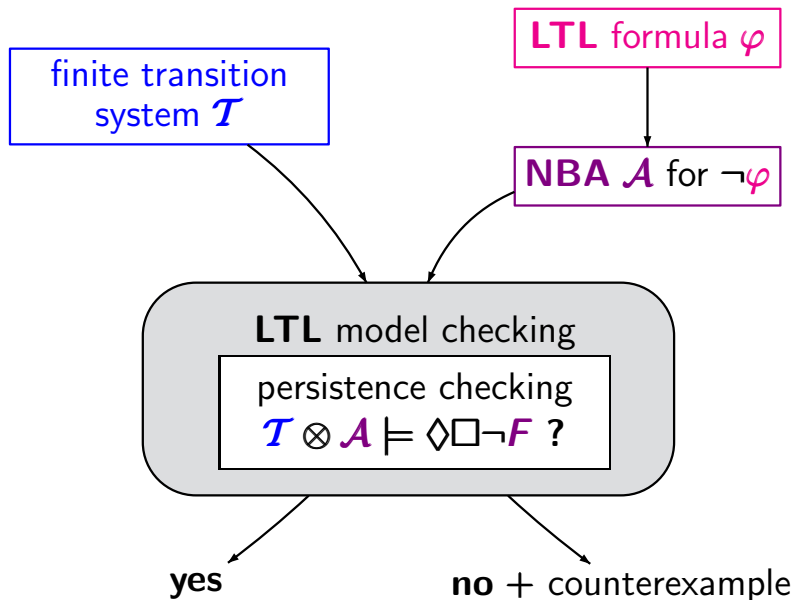
$\langle s_0, p_0 \rangle \dots \langle s_n, p_n \rangle \dots \langle s_n, p_n \rangle$

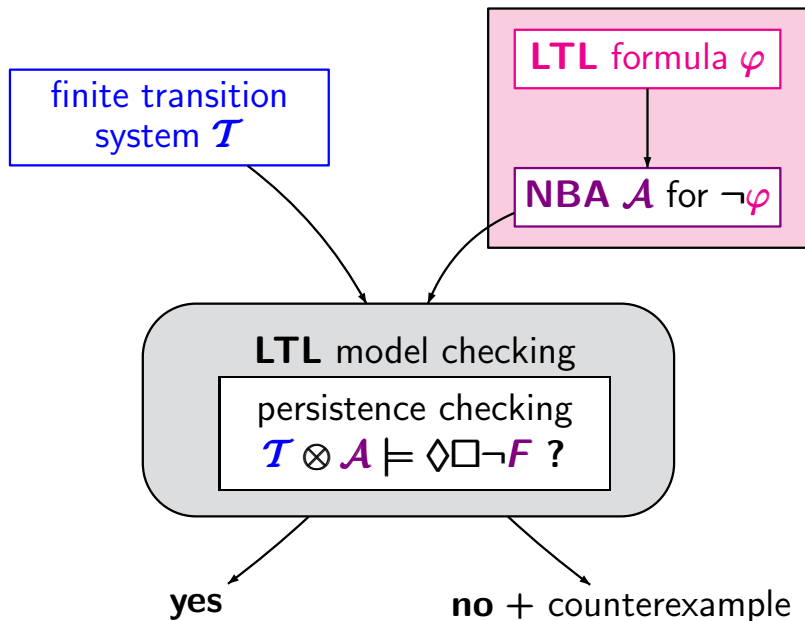
for $\mathcal{T} \otimes \mathcal{A}$ and $\diamond\Box\neg F$

return "no" and $s_0 \dots s_n \dots s_n$

persistence
checking
nested **DFS**

time complexity: $\mathcal{O}(\text{size}(\mathcal{T}) \cdot \text{size}(\mathcal{A}))$





For each **LTL** formula φ there is an **NBA** \mathcal{A} s.t.
 $\mathcal{L}_\omega(\mathcal{A}) = \text{Words}(\varphi)$

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LTL formula φ



NBA \mathcal{A} s.t.
 $\mathcal{L}_\omega(\mathcal{A}) = \text{Words}(\varphi)$

nondeterministic
Büchi automaton

For each **LTL** formula φ there is an **NBA** \mathcal{A} s.t.
 $\mathcal{L}_\omega(\mathcal{A}) = \text{Words}(\varphi)$

LTL formula φ

GNBA \mathcal{G} s.t.
 $\mathcal{L}_\omega(\mathcal{G}) = \text{Words}(\varphi)$

generalized NBA
several acceptance sets

NBA \mathcal{A} s.t.
 $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{G})$

nondeterministic
Büchi automaton
1 acceptance set

For each **LTL** formula φ there is an **NBA** \mathcal{A} s.t.
 $\mathcal{L}_\omega(\mathcal{A}) = \text{Words}(\varphi)$

LTL formula φ

GNBA \mathcal{G} s.t.
 $\mathcal{L}_\omega(\mathcal{G}) = \text{Words}(\varphi)$

NBA \mathcal{A} s.t.
 $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{G})$

generalized NBA
 k acceptance sets

k copies of \mathcal{G}

nondeterministic
Büchi automaton
 1 acceptance set

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA \mathcal{G}

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semantics of ...	encoding
propositional logic <i>true</i> , \neg , \wedge	
next \bigcirc	
until \mathbf{U}	

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA \mathcal{G}

semantics of ...	encoding
propositional logic <i>true</i> , \neg , \wedge	in the <i>states</i>
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semantics of ...	encoding
propositional logic <i>true</i> , \neg , \wedge	in the <i>states</i>
next \bigcirc	in the <i>transition relation</i>
until \mathbf{U}	

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA \mathcal{G}

semantics of ...	encoding
propositional logic <i>true</i> , \neg , \wedge	in the <i>states</i>
next \bigcirc	in the <i>transition relation</i>
until \mathbf{U}	via <i>expansion law</i>

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$$\psi_1 \mathbf{U} \psi_2 \equiv \psi_2 \vee (\psi_1 \wedge \bigcirc(\psi_1 \mathbf{U} \psi_2))$$

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA \mathcal{G}

semantics of ...	encoding
propositional logic <i>true</i> , \neg , \wedge	in the <i>states</i>
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encoded in
the *states*

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA \mathcal{G}

semantics of ...	encoding
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$$\psi_1 \mathbf{U} \psi_2 \equiv \psi_2 \vee (\psi_1 \wedge \bigcirc(\psi_1 \mathbf{U} \psi_2))$$

encoded in
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encoded in the
transition relation

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA \mathcal{G}

semantics of ...	encoding
propositional logic <i>true</i> , \neg , \wedge	in the <i>states</i>
next \bigcirc	in the <i>transition relation</i>
until \mathbf{U}	expansion law, <i>least fixed point</i>

$$\psi_1 \mathbf{U} \psi_2 \equiv \psi_2 \vee (\psi_1 \wedge \bigcirc(\psi_1 \mathbf{U} \psi_2))$$

encoded in
the *states*

encoded in the
transition relation

acceptance condition



LTL formula φ \rightsquigarrow GNBA \mathcal{G} for $Words(\varphi)$

LTL formula φ \rightsquigarrow GNBA \mathcal{G} for $Words(\varphi)$

states of $\mathcal{G} \hat{=} (\text{certain})$ sets of subformulas of φ

LTL formula $\varphi \rightsquigarrow$ GNBA \mathcal{G} for $Words(\varphi)$

states of $\mathcal{G} \hat{=} (certain)$ sets of subformulas of φ
s.t. each word $\sigma = A_0 A_1 A_2 \dots \in Words(\varphi)$ can be
extended to an accepting run $B_0 B_1 B_2 \dots$ in \mathcal{G}

LTL formula $\varphi \rightsquigarrow$ GNBA \mathcal{G} for $Words(\varphi)$

states of $\mathcal{G} \hat{=} (certain)$ sets of subformulas of φ
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$A_0 A_1 A_2 A_3 \dots \in Words(\varphi)$

LTL formula $\varphi \rightsquigarrow$ GNBA \mathcal{G} for $Words(\varphi)$

states of $\mathcal{G} \hat{=} (\text{certain})$ sets of subformulas of φ
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$$\downarrow \ \downarrow \ \downarrow \ \downarrow$$

$$B_0 \ B_1 \ B_2 \ B_3 \ \dots \text{ accepting run}$$

where $B_i = \{ \psi \in cl(\varphi) : A_i A_{i+1} A_{i+2} \dots \models \psi \}$

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set of subformulas of φ and their negations

LTL formula $\varphi \rightsquigarrow$ GNBA \mathcal{G} for $Words(\varphi)$

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Example: $\varphi = a U(\neg a \wedge b)$

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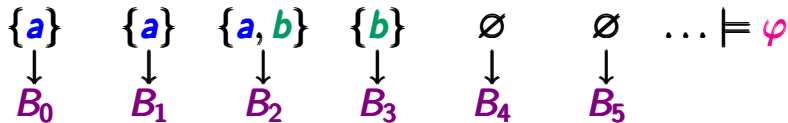
Example: $\varphi = a U (\neg a \wedge b)$

$\{a\}$ $\{a\}$ $\{a, b\}$ $\{b\}$ \emptyset \emptyset $\dots \models \varphi$

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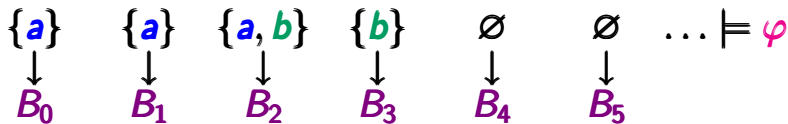
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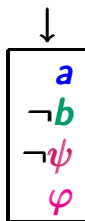
where the B_i 's are subsets of
 $\{a, \neg a, b, \neg b, \psi, \neg\psi, \varphi, \neg\varphi\}$

LTL formula $\varphi \rightsquigarrow$ GNBA \mathcal{G} for $Words(\varphi)$

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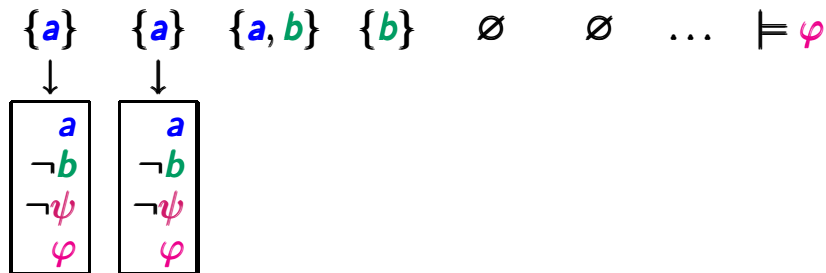


just for better readability:
 tuple rather than set notation

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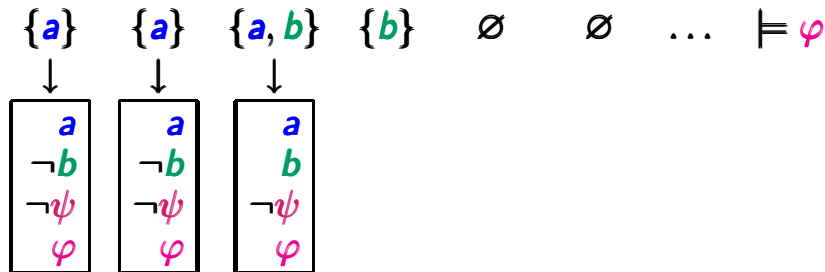
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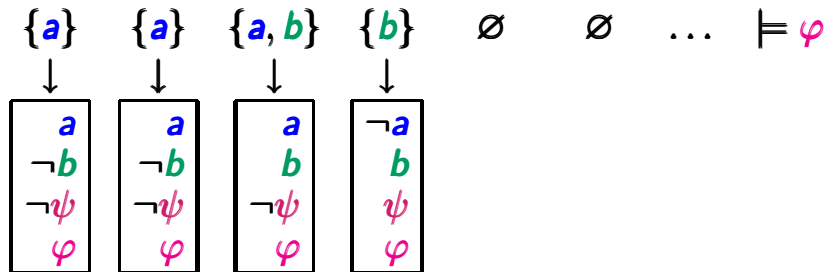
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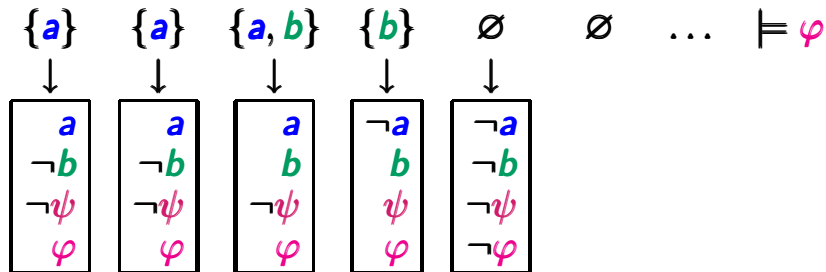
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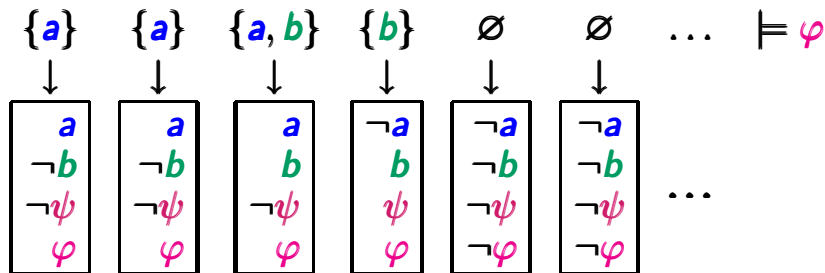
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Let φ be an LTL formula. Then:

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Example: if $\varphi' = \Box a = \neg\Diamond\neg a = \neg(true \cup \neg a)$

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Example: if $\varphi' = \Box a = \neg\Diamond\neg a = \neg(true \cup \neg a)$ then

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Let $B \subseteq cl(\varphi)$. B is called elementary if:

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if $\psi \in B$ then $\neg\psi \notin B$

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if $\psi \in cl(\varphi) \setminus B$ then $\neg\psi \in B$

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if $\psi_1 \wedge \psi_2 \in B$ then $\neg\psi_1 \notin B$ and $\neg\psi_2 \notin B$

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if $false \in cl(\varphi)$ then $false \notin B$

(2) B is maximal consistent

if $\psi \in cl(\varphi) \setminus B$ then $\neg\psi \in B$

(3) B is locally consistent with respect to until \mathbf{U} :

if $\psi_1 \mathbf{U} \psi_2 \in B$ and $\neg\psi_2 \in B$ then $\neg\psi_1 \notin B$

Let $B \subseteq cl(\varphi)$. B is called elementary if:

(1) B is consistent w.r.t. propositional logic

if $\psi \in B$ then $\neg\psi \notin B$

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if $false \in cl(\varphi)$ then $false \notin B$

(2) B is maximal consistent

if $\psi \in cl(\varphi) \setminus B$ then $\neg\psi \in B$

(3) B is locally consistent with respect to until U :

if $\psi_1 U \psi_2 \in B$ and $\neg\psi_2 \in B$ then $\neg\psi_1 \notin B$

if $\psi_2 \in B$ and $\psi_1 U \psi_2 \in cl(\varphi)$ then $\neg(\psi_1 U \psi_2) \notin B$

$B \subseteq cl(\varphi)$ is elementary iff:

- (i) B is maximal consistent w.r.t. prop. logic, i.e., if $\psi, \psi_1 \wedge \psi_2 \in cl(\varphi)$ then:

$\psi \notin B$	iff	$\neg\psi \in B$
$\psi_1 \wedge \psi_2 \in B$	iff	$\psi_1 \in B$ and $\psi_2 \in B$
$true \in cl(\varphi)$	implies	$true \in B$

- (ii) B is locally consistent with respect to until \mathbf{U} , i.e., if $\psi_1 \mathbf{U} \psi_2 \in cl(\varphi)$ then:

if $\psi_1 \mathbf{U} \psi_2 \in B$ and $\psi_2 \notin B$	then $\psi_1 \in B$
if $\psi_2 \in B$	then $\psi_1 \mathbf{U} \psi_2 \in B$

Elementary or not?

LTLMC3.2-49

Let $\varphi = a \text{ U } (\neg a \wedge b)$.

$B_1 = \{a, b, \neg a \wedge b, \varphi\}$

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not elementary
propositional inconsistent

Let $\varphi = a \mathbf{U}(\neg a \wedge b)$.

$$B_1 = \{a, b, \neg a \wedge b, \varphi\}$$

not elementary
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$$B_2 = \{\neg a, b, \varphi\}$$

Let $\varphi = a \vee (\neg a \wedge b)$.

$$B_1 = \{a, b, \neg a \wedge b, \varphi\}$$

not elementary
propositional inconsistent

$$B_2 = \{\neg a, b, \varphi\}$$

not elementary, not maximal

as $\neg a \wedge b \notin B_2$

$\neg(\neg a \wedge b) \notin B_2$

Let $\varphi = a \text{ U } (\neg a \wedge b)$.

$$B_1 = \{a, b, \neg a \wedge b, \varphi\}$$

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not elementary, not maximal

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$$\neg(\neg a \wedge b) \notin B_2$$

$$B_3 = \{\neg a, b, \neg a \wedge b, \neg \varphi\}$$

Let $\varphi = a \mathbf{U} (\neg a \wedge b)$.

$$B_1 = \{a, b, \neg a \wedge b, \varphi\}$$

not elementary
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$$B_2 = \{\neg a, b, \varphi\}$$

not elementary, not maximal
as $\neg a \wedge b \notin B_2$
 $\neg(\neg a \wedge b) \notin B_2$

$$B_3 = \{\neg a, b, \neg a \wedge b, \neg \varphi\}$$

not elementary
not locally consistent for \mathbf{U}

Elementary or not?

Let $\varphi = a \mathbf{U} (\neg a \wedge b)$.

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$$B_2 = \{\neg a, b, \varphi\}$$

not elementary, not maximal
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not elementary
not locally consistent for \mathbf{U}

$$B_4 = \{\neg a, \neg b, \neg(\neg a \wedge b), \neg \varphi\}$$

Let $\varphi = a \mathbf{U} (\neg a \wedge b)$.

$$B_1 = \{a, b, \neg a \wedge b, \varphi\}$$

not elementary
propositional inconsistent

$$B_2 = \{\neg a, b, \varphi\}$$

not elementary, not maximal
as $\neg a \wedge b \notin B_2$
 $\neg(\neg a \wedge b) \notin B_2$

$$B_3 = \{\neg a, b, \neg a \wedge b, \neg \varphi\}$$

not elementary
not locally consistent for \mathbf{U}

$$B_4 = \{\neg a, \neg b, \neg(\neg a \wedge b), \neg \varphi\} \quad \text{elementary}$$

closure $cl(\varphi)$:

- set of all subformulas of φ and their negations
- ψ and $\neg\neg\psi$ are identified

elementary formula-sets: subsets B of $cl(\varphi)$

- maximal consistent w.r.t. propositional logic
- locally consistent w.r.t. \mathbf{U}

For $\varphi = a \mathbf{U} (\neg a \wedge b)$, the elementary sets are:

$$\begin{array}{ll} \{ a, b, \neg(\neg a \wedge b), \varphi \} & \{ a, b, \neg(\neg a \wedge b), \neg\varphi \} \\ \{ a, \neg b, \neg(\neg a \wedge b), \varphi \} & \{ a, \neg b, \neg(\neg a \wedge b), \neg\varphi \} \\ \{ \neg a, b, \neg a \wedge b, \varphi \} & \{ \neg a, \neg b, \neg(\neg a \wedge b), \neg\varphi \} \end{array}$$

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA \mathcal{G} :

semantics of ...	encoding
propositional logic <i>true</i> , \neg , \wedge	in the <i>states</i>
next \bigcirc	in the <i>transition relation</i>
until \mathbf{U}	expansion law, least fixed point

$$\psi_1 \mathbf{U} \psi_2 \equiv \psi_2 \vee (\psi_1 \wedge \bigcirc(\psi_1 \mathbf{U} \psi_2))$$

encoded in
the *states*

encoded in the
transition relation

acceptance condition

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA \mathcal{G} :

semantics of ...	encoding
propositional logic $true, \neg, \wedge$	in the states ← elementary formula sets
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\uparrow

elementary formula sets

encoded in the **transition relation**

acceptance condition

$$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$$

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state space: $Q = \{B \subseteq cl(\varphi) : B \text{ is elementary}\}$

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initial states: $Q_0 = \{B \in Q : \varphi \in B\}$

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if $A \neq B \cap AP$ then $\delta(B, A) = \emptyset$

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if $A = B \cap AP$ then $\delta(B, A) = \text{set of all } B' \in Q \text{ s.t.}$

$$\bigcirc \psi \in B \text{ iff } \psi \in B'$$

$$\psi_1 \mathbf{U} \psi_2 \in B \text{ iff } (\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \mathbf{U} \psi_2 \in B')$$

$$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$$

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acceptance set $\mathcal{F} = \{F_{\psi_1 \mathbf{U} \psi_2} : \psi_1 \mathbf{U} \psi_2 \in cl(\varphi)\}$

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where $F_{\psi_1 \mathbf{U} \psi_2} = \{B \in Q : \psi_1 \mathbf{U} \psi_2 \notin B \vee \psi_2 \in B\}$

Example: GNBA for $\varphi = \bigcirc a$

LTLMC3.2-52

Example: GNBA for $\varphi = \bigcirc a$

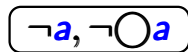
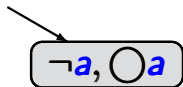
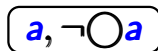
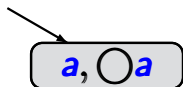
LTLMC3.2-52

$a, \bigcirc a$

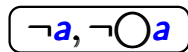
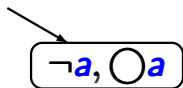
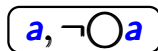
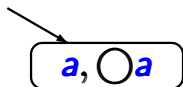
$a, \neg \bigcirc a$

$\neg a, \bigcirc a$

$\neg a, \neg \bigcirc a$



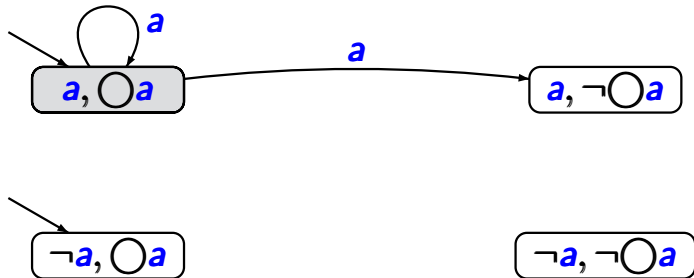
initial states: formula-sets B with $\bigcirc a \in B$



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transition relation:

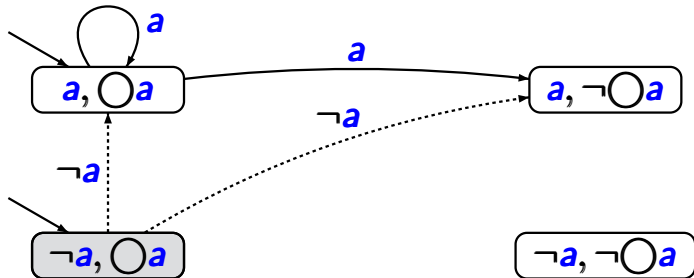
if $\bigcirc a \in B$ then $\delta(B, B \cap \{a\}) = \{B' : a \in B'\}$



initial states: formula-sets B with $\bigcirc a \in B$

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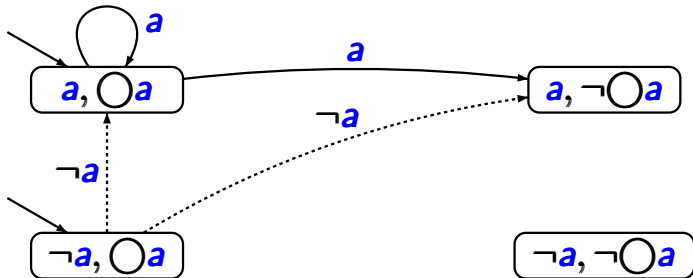
$$\text{if } \bigcirc a \in B \text{ then } \delta(B, B \cap \{a\}) = \{B' : a \in B'\}$$



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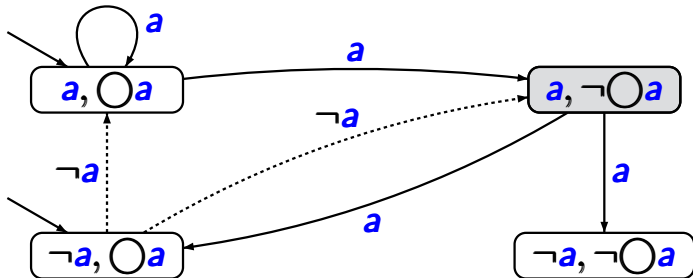


initial states: formula-sets B with $\bigcirc a \in B$

transition relation:

if $\bigcirc a \in B$ then $\delta(B, B \cap \{a\}) = \{B' : a \in B'\}$

if $\bigcirc a \notin B$ then $\delta(B, B \cap \{a\}) = \{B' : a \notin B'\}$

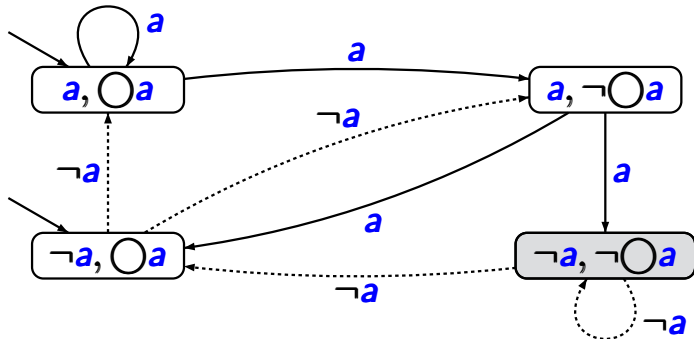


initial states: formula-sets B with $\bigcirc a \in B$

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if $\bigcirc a \in B$ then $\delta(B, B \cap \{a\}) = \{B' : a \in B'\}$

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initial states: formula-sets B with $\bigcirc a \in B$

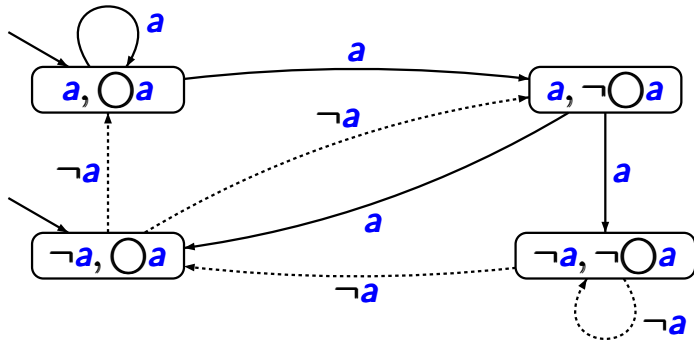
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Example: GNBA for $\varphi = \bigcirc a$

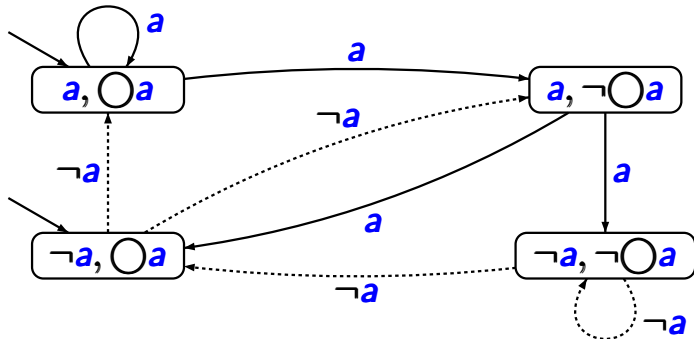
LTLMC3.2-53



set of acceptance sets:

Example: GNBA for $\varphi = \bigcirc a$

LTLMC3.2-53

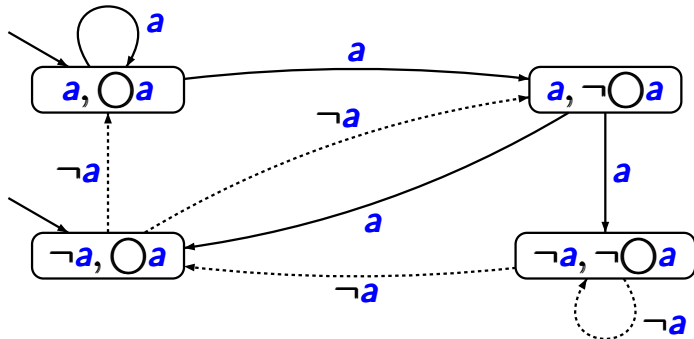


set of acceptance sets: $\mathcal{F} = \emptyset$

hence: all words having an **infinite run** are accepted

Example: GNBA for $\varphi = \bigcirc a$

LTLMC3.2-53

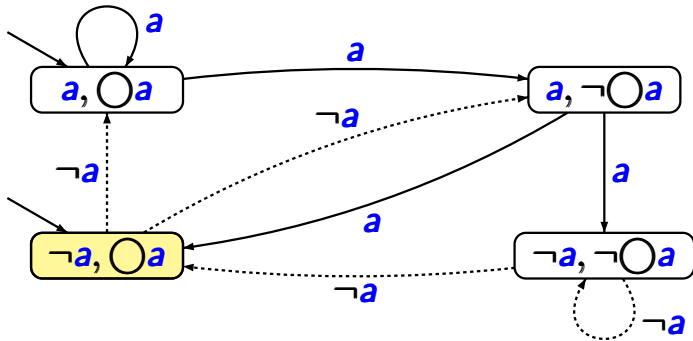


set of acceptance sets: $\mathcal{F} = \emptyset$

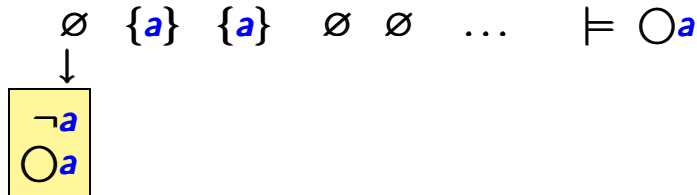
$\emptyset \quad \{a\} \quad \{a\} \quad \emptyset \quad \emptyset \quad \dots \quad \models \bigcirc a$

Example: GNBA for $\varphi = \bigcirc a$

LTLMC3.2-53

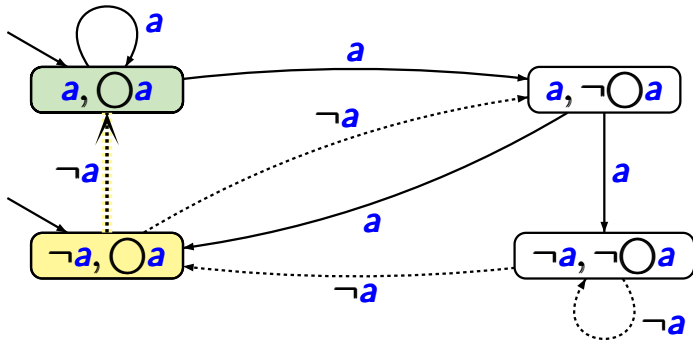


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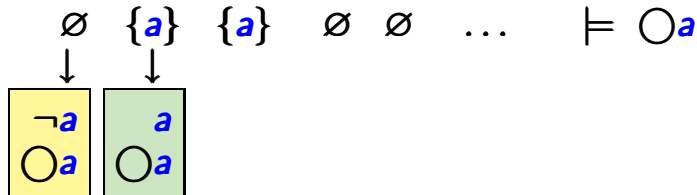


Example: GNBA for $\varphi = \bigcirc a$

LTLMC3.2-53

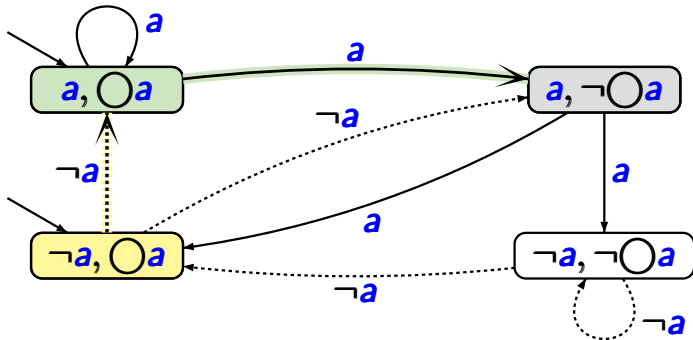


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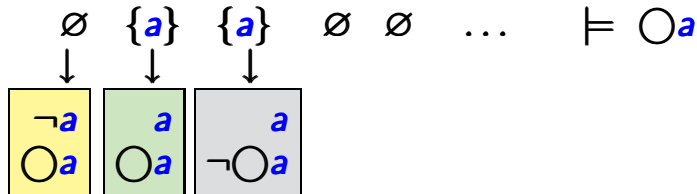


Example: GNBA for $\varphi = \bigcirc a$

LTLMC3.2-53

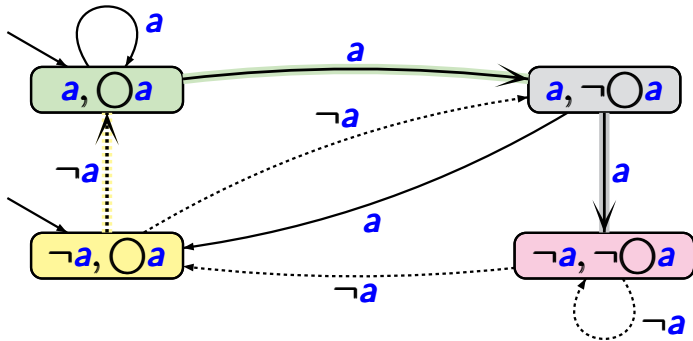


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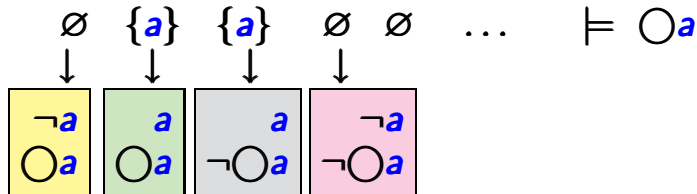


Example: GNBA for $\varphi = \bigcirc a$

LTLMC3.2-53

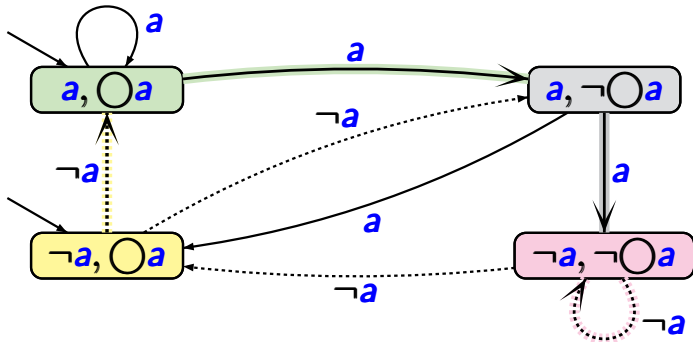


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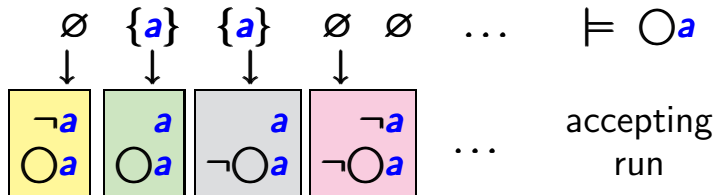


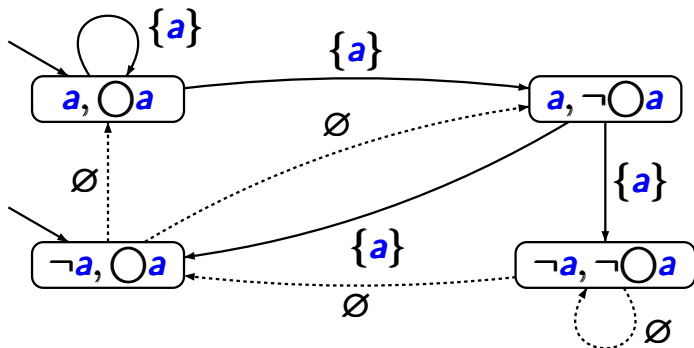
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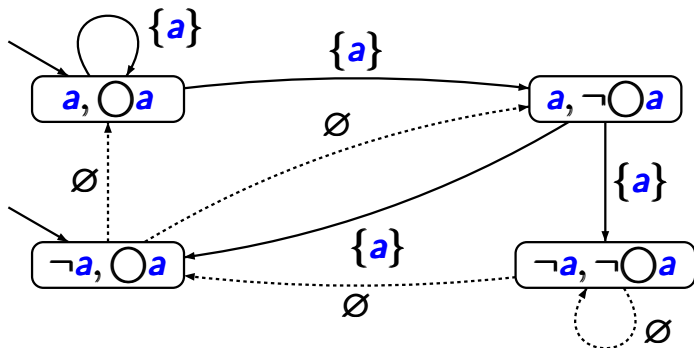


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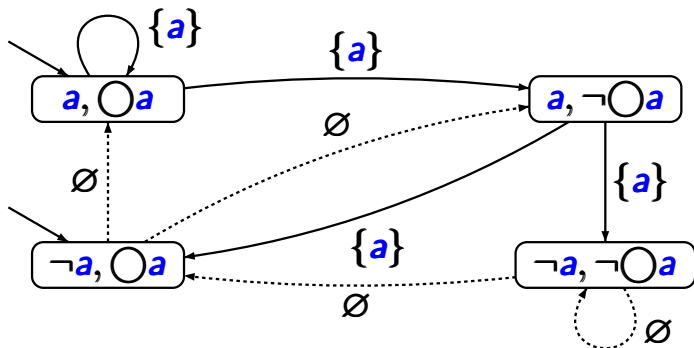


for all words $\sigma = A_0 A_1 A_2 A_3 \dots \in \mathcal{L}_\omega(\mathcal{G})$: $A_1 = \{a\}$



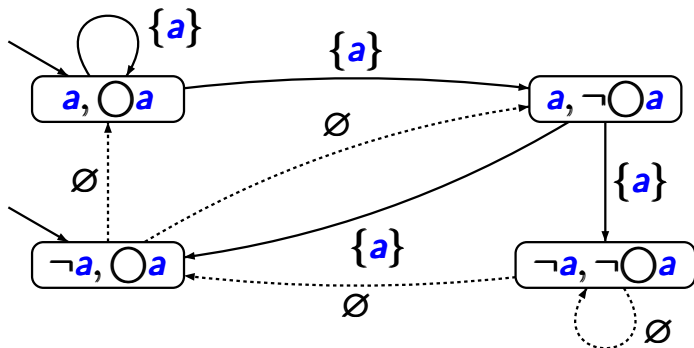
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proof:



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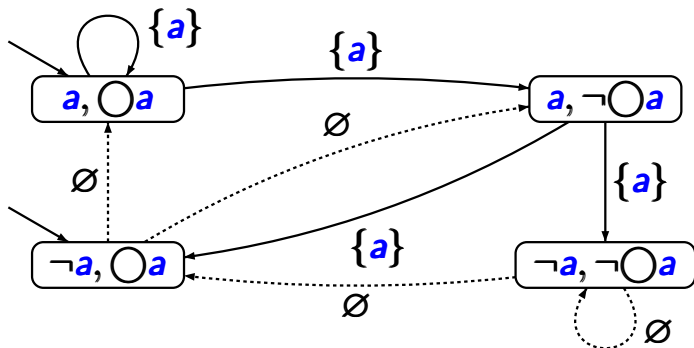
proof: Let $B_0 B_1 B_2 \dots$ be an accepting run for σ .



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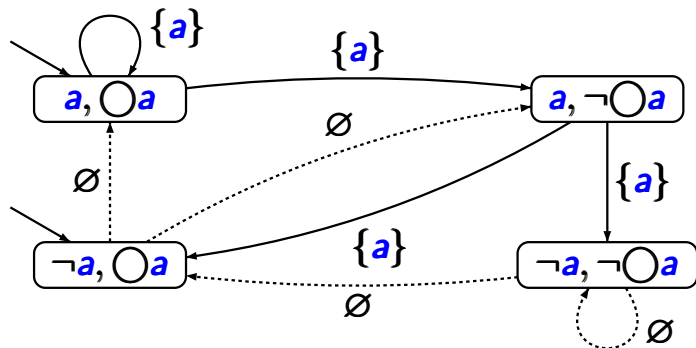
$\implies \bigcirc a \in B_0$



for all words $\sigma = A_0 A_1 A_2 A_3 \dots \in \mathcal{L}_\omega(\mathcal{G})$: $A_1 = \{a\}$

proof: Let $B_0 B_1 B_2 \dots$ be an accepting run for σ .

$\implies \bigcirc a \in B_0$ and therefore $a \in B_1$

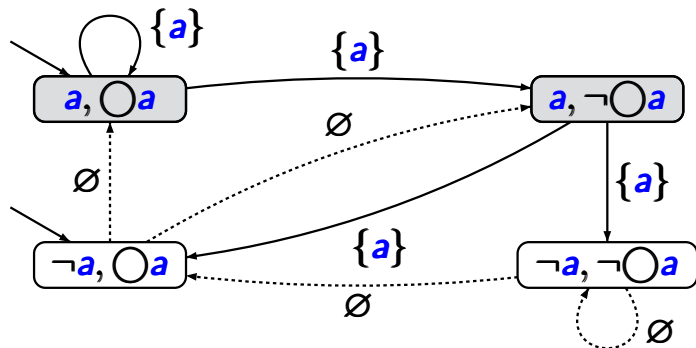


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\implies the outgoing edges of B_1 have label $\{a\}$



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\implies the outgoing edges of B_1 have label $\{a\}$

$\implies \{a\} = B_1 \cap AP = A_1$

Example: GNBA for $\varphi = aU b$

LTLMC3.2-54

$a, b, a \cup b$

$\neg a, \neg b, \neg(a \cup b)$

$a, \neg b, a \cup b$

$a, \neg b, \neg(a \cup b)$

$\neg a, b, a \cup b$

locally inconsistent: $\{a, b, \neg(a \cup b)\}$

$\{\neg a, b, \neg(a \cup b)\}$

$\{\neg a, \neg b, a \cup b\}$

$a, b, a \mathbf{U} b$

$\neg a, \neg b, \neg(a \mathbf{U} b)$

$a, \neg b, a \mathbf{U} b$

$a, \neg b, \neg(a \mathbf{U} b)$

$\neg a, b, a \mathbf{U} b$

initial states:

B with $\varphi = a \mathbf{U} b \in B$

→ $a, b, a \mathbf{U} b$

$\neg a, \neg b, \neg(a \mathbf{U} b)$

→ $a, \neg b, a \mathbf{U} b$

$a, \neg b, \neg(a \mathbf{U} b)$

→ $\neg a, b, a \mathbf{U} b$

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B with $\varphi = a \mathbf{U} b \in B$

→ $a, b, a \mathbf{U} b$

$\neg a, \neg b, \neg(a \mathbf{U} b)$

→ $a, \neg b, a \mathbf{U} b$

$a, \neg b, \neg(a \mathbf{U} b)$

→ $\neg a, b, a \mathbf{U} b$

initial states: B with $\varphi = a \mathbf{U} b \in B$

acceptance condition: just one set of accept states

$F =$ set of all B with $\varphi \notin B$ or $b \in B$

$\longrightarrow a, b, a \cup b$

$\neg a, \neg b, \neg(a \cup b)$

$\longrightarrow a, \neg b, a \cup b$

$a, \neg b, \neg(a \cup b)$

$\longrightarrow \neg a, b, a \cup b$

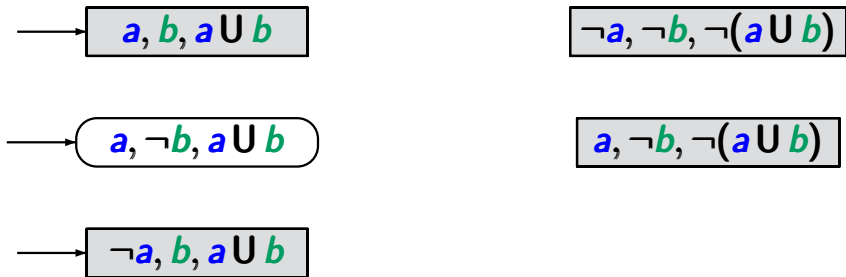
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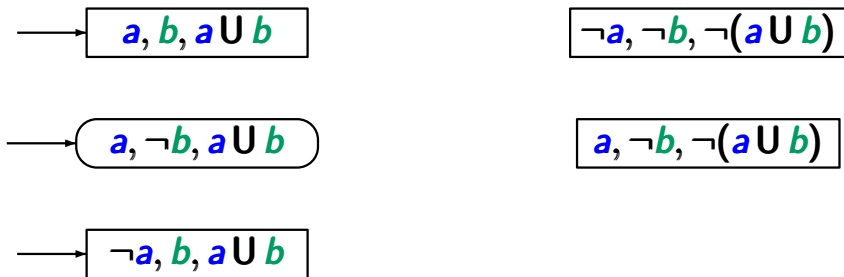


initial states:

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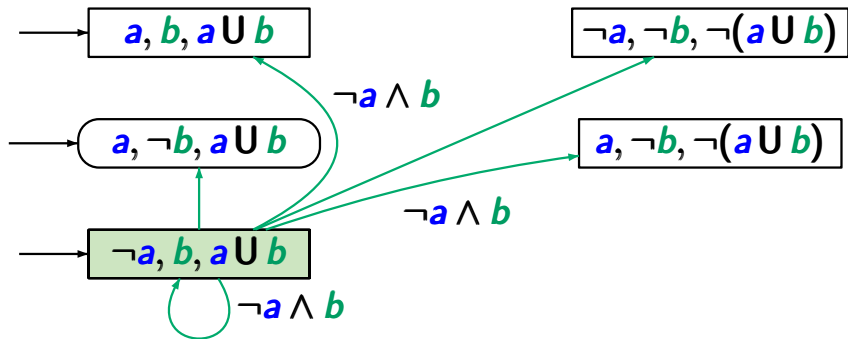
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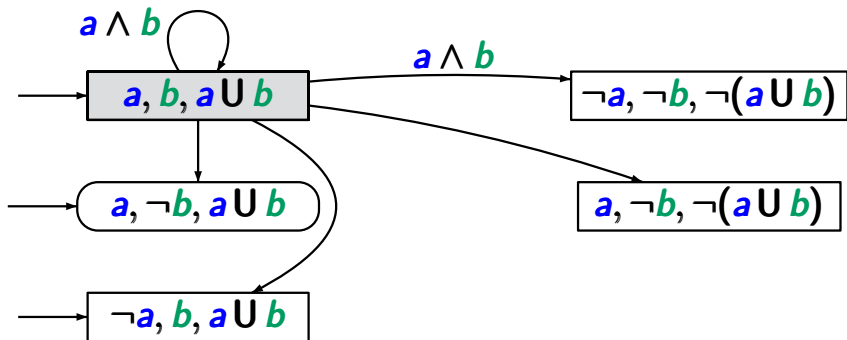
transition relation: $B' \in \delta(B, B \cap AP)$ iff

$$a \cup b \in B \iff (b \in B \vee (a \in B \wedge a \cup b \in B'))$$



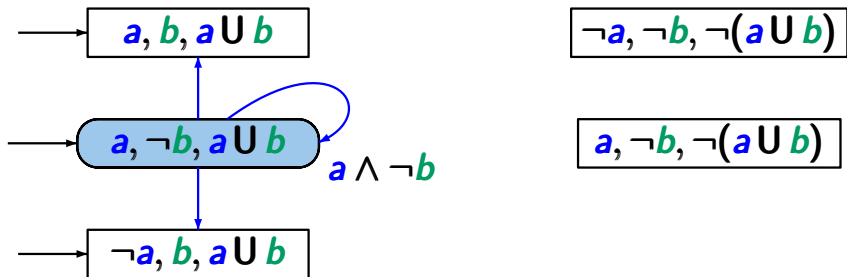
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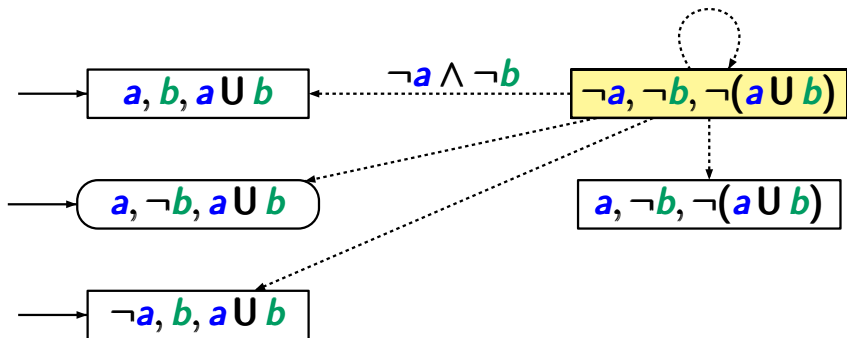
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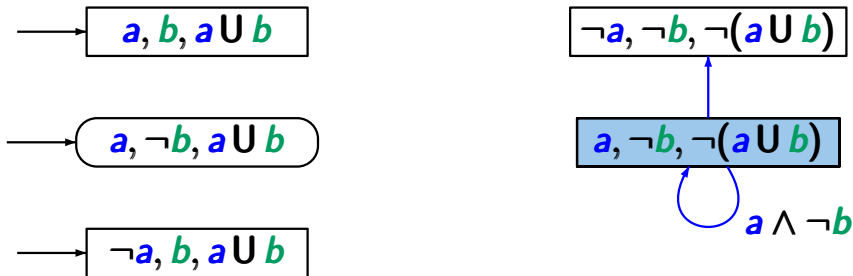
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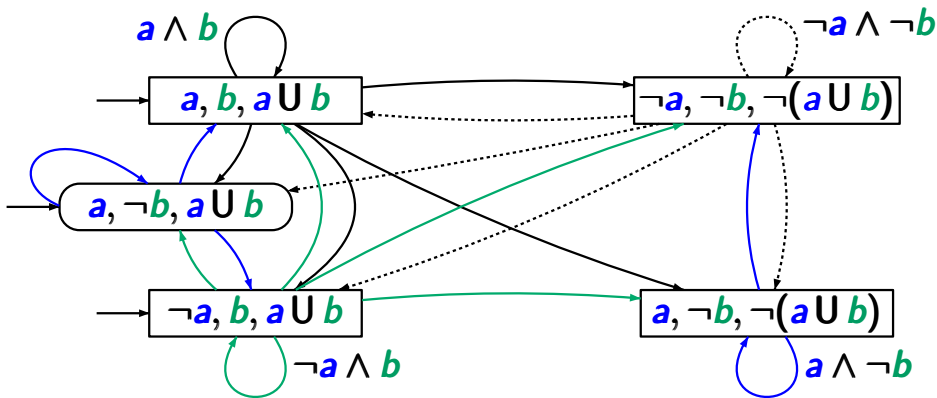


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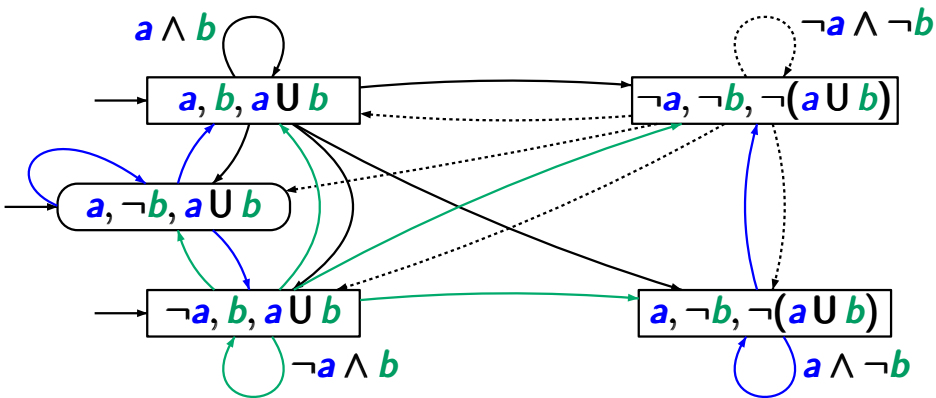
Example: (G)NBA for $\varphi = a \cup b$

LTLMC3.2-55



Example: (G)NBA for $\varphi = aU b$

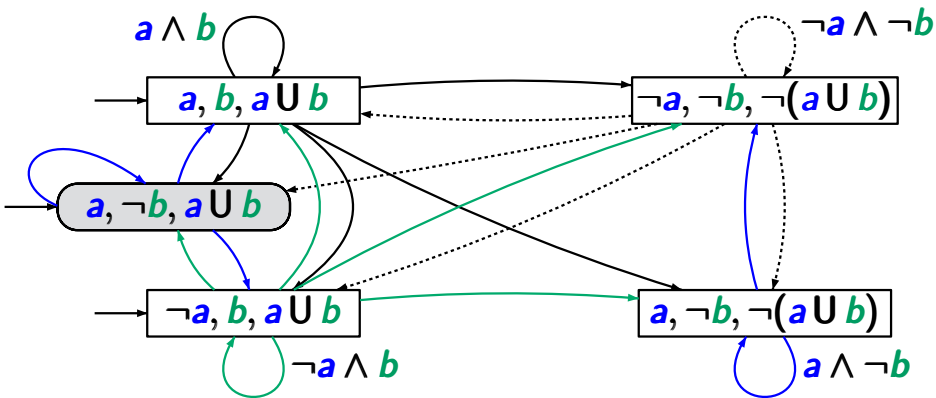
LTLMC3.2-55



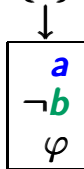
$\{a\}$ $\{a\}$ $\{a, b\}$ \emptyset \emptyset \emptyset ... $\models aU b$

Example: (G)NBA for $\varphi = a \cup b$

LTLMC3.2-55

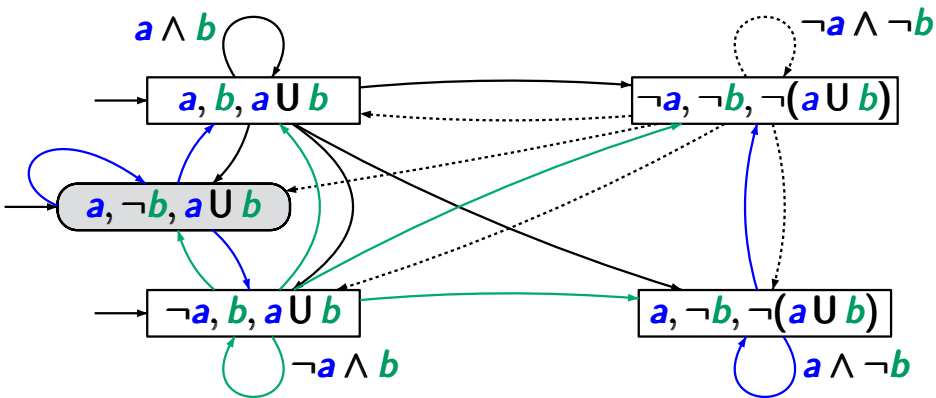


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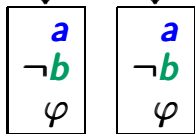


Example: (G)NBA for $\varphi = aU b$

LTLMC3.2-55

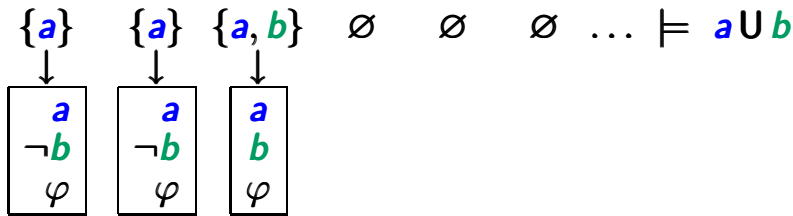
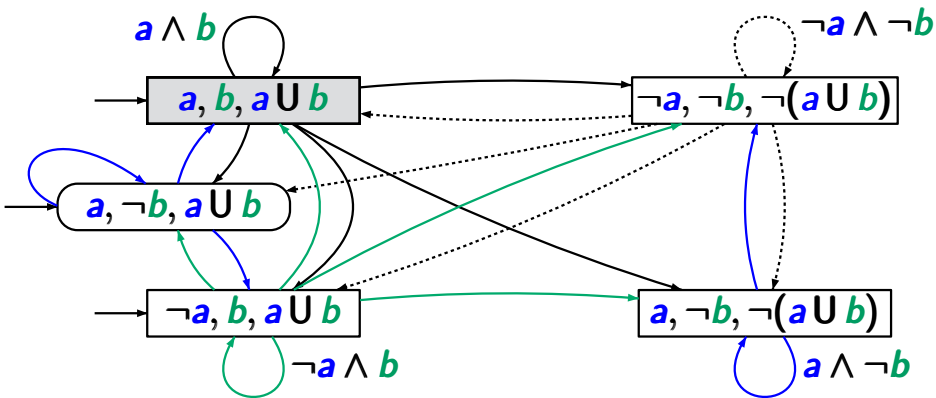


$\{a\} \quad \{a\} \quad \{a, b\} \quad \emptyset \quad \emptyset \quad \emptyset \quad \dots \models aU b$



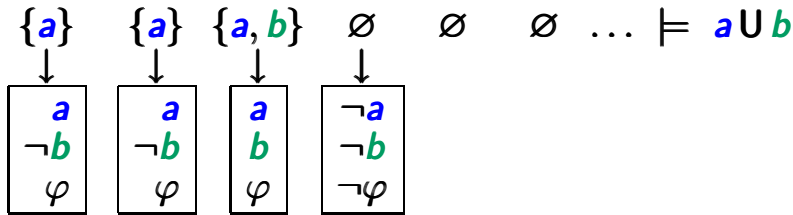
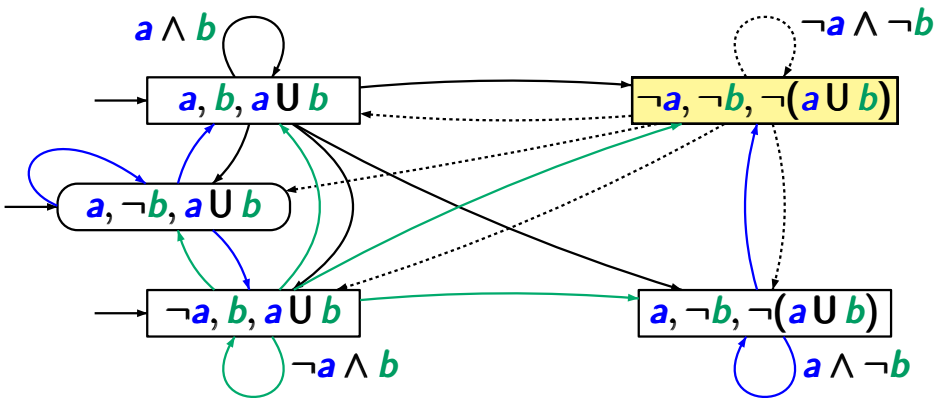
Example: (G)NBA for $\varphi = aU b$

LTLMC3.2-55



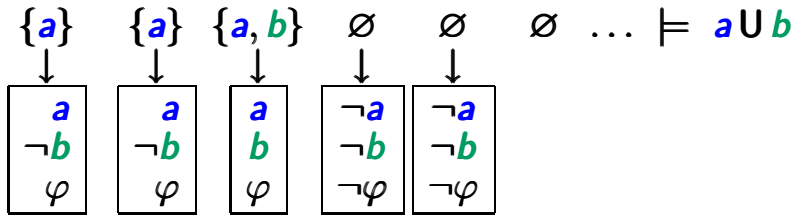
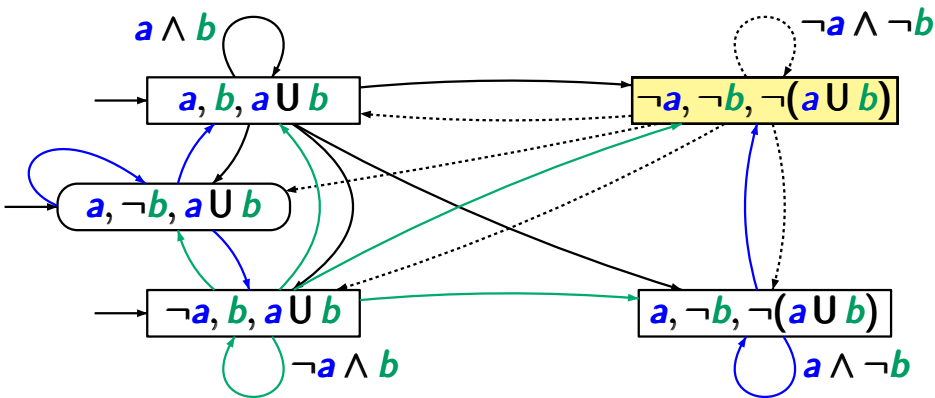
Example: (G)NBA for $\varphi = aU b$

LTLMC3.2-55



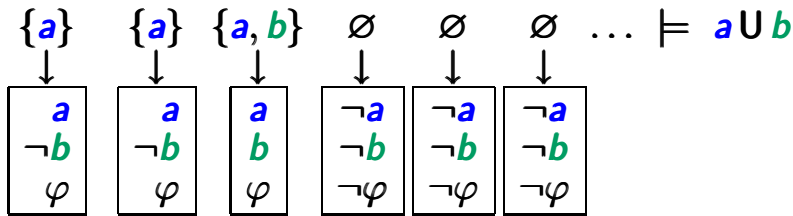
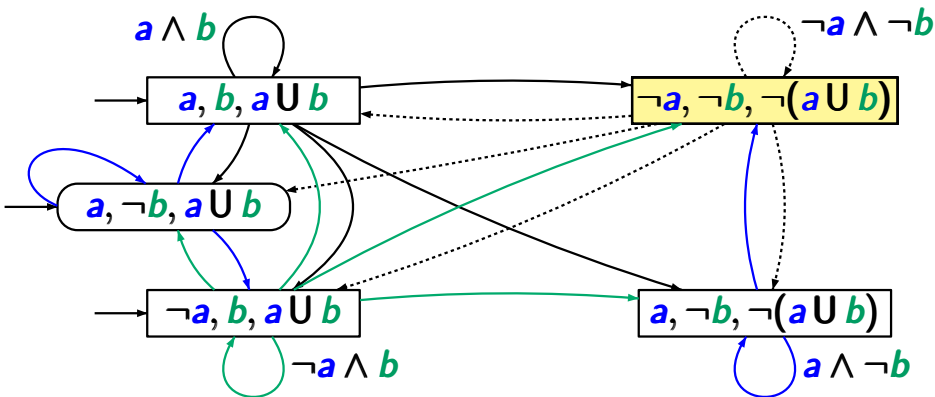
Example: (G)NBA for $\varphi = aU b$

LTLMC3.2-55



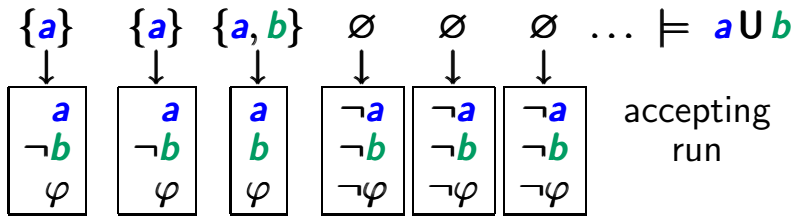
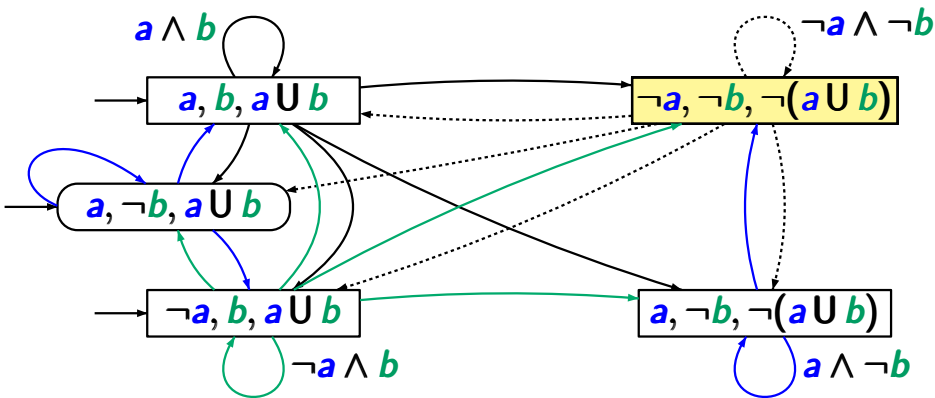
Example: (G)NBA for $\varphi = a \cup b$

LTLMC3.2-55



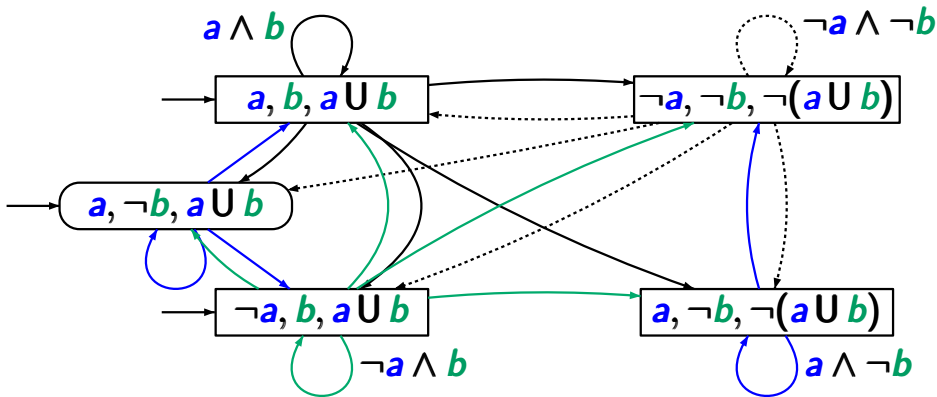
Example: (G)NBA for $\varphi = a \cup b$

LTLMC3.2-55



Example: (G)NBA for $\varphi = a \cup b$

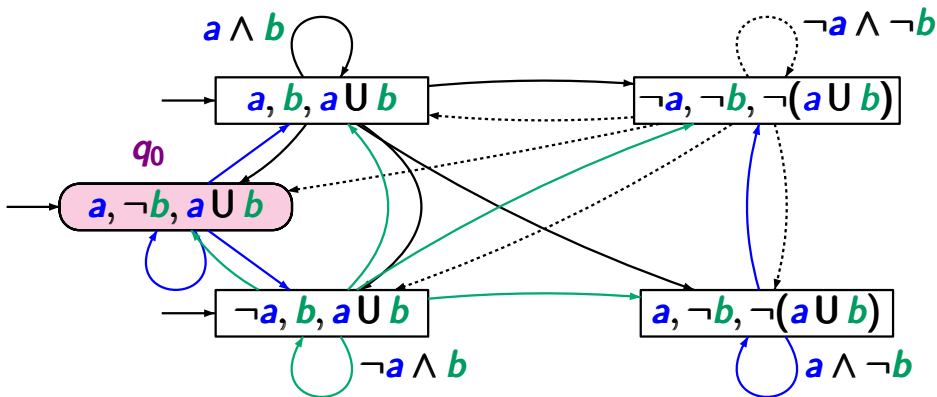
LTLMC3.2-56



$\{a\} \{a\} \{a\} \{a\} \dots \not\models \varphi$

Example: (G)NBA for $\varphi = a \cup b$

LTLMC3.2-56

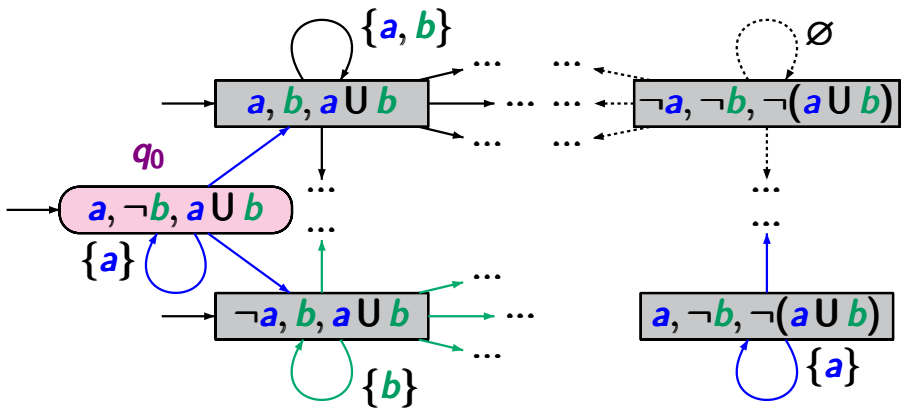


$\{a\} \{a\} \{a\} \{a\} \dots \not\models \varphi$

only 1 infinite run: $q_0 q_0 q_0 \dots$

Example: (G)NBA for $\varphi = a U b$

LTLMC3.2-56

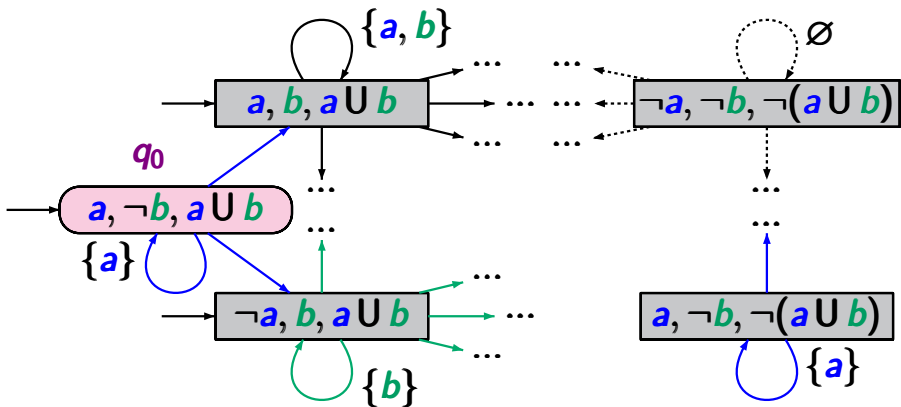


$\{a\} \{a\} \{a\} \{a\} \dots \not\models \varphi$

only 1 infinite run: $q_0 q_0 q_0 \dots$

Example: (G)NBA for $\varphi = a U b$

LTLMC3.2-56



$\{a\} \{a\} \{a\} \{a\} \dots \not\models \varphi$

only 1 infinite run: $q_0 q_0 q_0 \dots$ not accepting

$$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$$

state space: $Q = \{B \subseteq cl(\varphi) : B \text{ is elementary}\}$

initial states: $Q_0 = \{B \in Q : \varphi \in B\}$

transition relation: for $B \in Q$ and $A \in 2^{AP}$:

if $A \neq B \cap AP$ then $\delta(B, A) = \emptyset$

if $A = B \cap AP$ then $\delta(B, A) = \text{set of all } B' \in Q \text{ s.t.}$

$$\bigcirc \psi \in B \text{ iff } \psi \in B'$$

$$\psi_1 \mathbf{U} \psi_2 \in B \text{ iff } (\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \mathbf{U} \psi_2 \in B')$$

acceptance set $\mathcal{F} = \{F_{\psi_1 \mathbf{U} \psi_2} : \psi_1 \mathbf{U} \psi_2 \in cl(\varphi)\}$

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.... of the construction LTL formula $\varphi \rightsquigarrow$ GNBA \mathcal{G}

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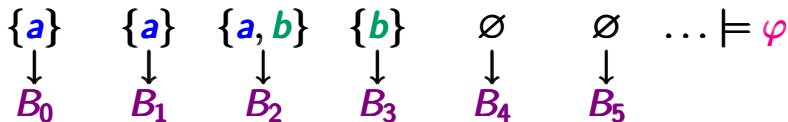
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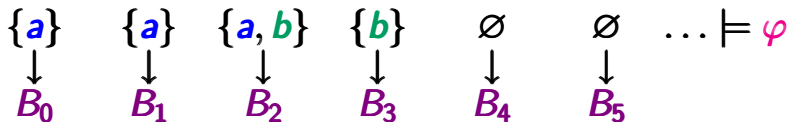
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Example: $\varphi = a U(\neg a \wedge b)$ $\psi = \neg a \wedge b$



where the B_i 's are states in \mathcal{G} , i.e., elementary subsets of $\{a, \neg a, b, \neg b, \psi, \neg \psi, \varphi, \neg \varphi\}$

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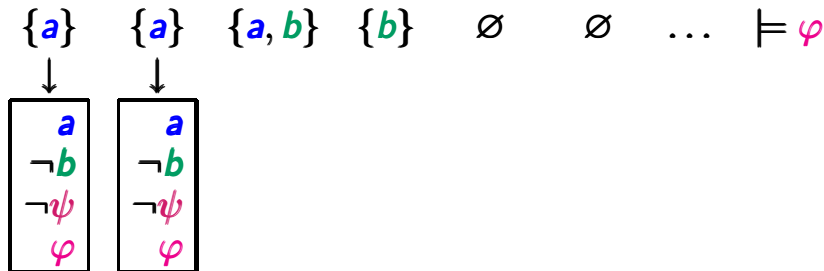
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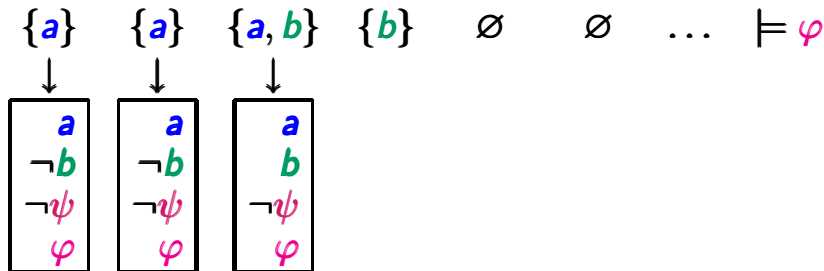
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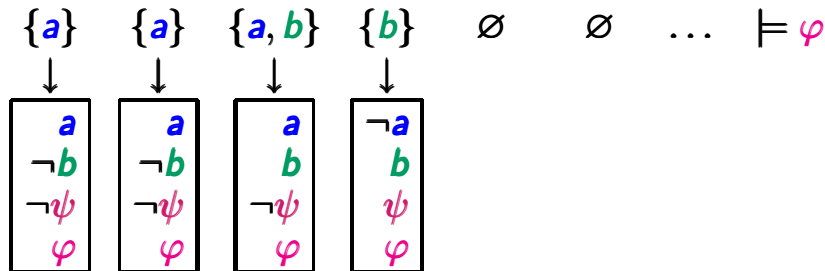
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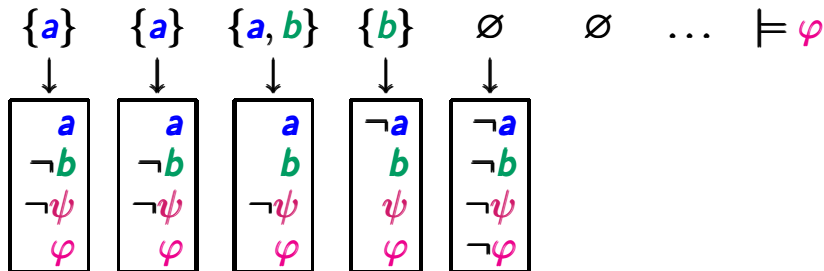
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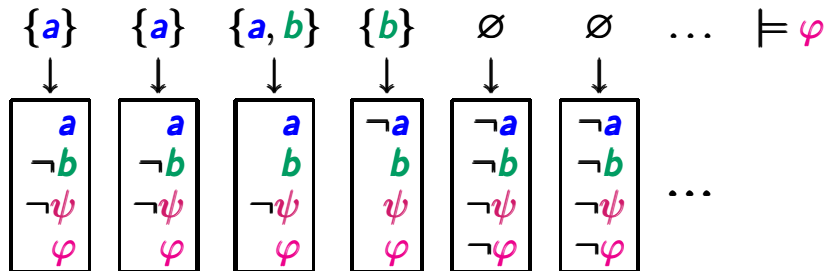
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$$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$$

state space: $Q = \{B \subseteq cl(\varphi) : B \text{ is elementary}\}$

initial states: $Q_0 = \{B \in Q : \varphi \in B\}$

transition relation: for $B \in Q$ and $A \in 2^{AP}$:

if $A \neq B \cap AP$ then $\delta(B, A) = \emptyset$

if $A = B \cap AP$ then $\delta(B, A) = \text{set of all } B' \in Q \text{ s.t.}$

$$\bigcirc \psi \in B \text{ iff } \psi \in B'$$

$$\psi_1 \mathbf{U} \psi_2 \in B \text{ iff } (\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \mathbf{U} \psi_2 \in B')$$

acceptance set $\mathcal{F} = \{F_{\psi_1 \mathbf{U} \psi_2} : \psi_1 \mathbf{U} \psi_2 \in cl(\varphi)\}$

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$B \subseteq cl(\varphi)$ is elementary iff:

- (i) B is maximal consistent w.r.t. prop. logic, i.e., if $\psi, \psi_1 \wedge \psi_2 \in cl(\varphi)$ then:

$$\begin{array}{l} \psi \notin B \quad \text{iff} \quad \neg\psi \in B \\ \psi_1 \wedge \psi_2 \in B \quad \text{iff} \quad \psi_1 \in B \text{ and } \psi_2 \in B \\ \text{true} \in cl(\varphi) \quad \text{implies} \quad \text{true} \in B \end{array}$$

- (ii) B is locally consistent with respect to until \mathbf{U} , i.e., if $\psi_1 \mathbf{U} \psi_2 \in cl(\varphi)$ then:

$$\begin{array}{l} \text{if } \psi_1 \mathbf{U} \psi_2 \in B \text{ and } \psi_2 \notin B \text{ then } \psi_1 \in B \\ \text{if } \psi_2 \in B \text{ then } \psi_1 \mathbf{U} \psi_2 \in B \end{array}$$

Let φ be an LTL-formula and $\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$ be the constructed GNBA.

Claim: $Words(\varphi) = \mathcal{L}_\omega(\mathcal{G})$

“ \subseteq ” show: each infinite word $A_0 A_1 A_2 \dots \in (2^{AP})^\omega$

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$$\forall F \in \mathcal{F} \exists j \geq 0. B_j \in F$$

then for all formulas $\psi \in \text{cl}(\varphi)$:

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as $B_0 \in Q_0$

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$$\forall F \in \mathcal{F} \exists j \geq 0. B_j \in F \quad (*)$$

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and $(*)$ holds

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Proof by structural induction on ψ

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Proof by structural induction on ψ

base of induction:

$$\psi = \text{true}$$

$$\psi = a \in AP$$

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Proof by structural induction on ψ

base of induction:

$$\psi = \text{true}$$

$$\psi = a \in AP$$

induction step:

$$\psi = \neg \psi'$$

$$\psi = \psi_1 \wedge \psi_2$$

$$\psi = \bigcirc \psi'$$

$$\psi = \psi_1 \cup \psi_2$$

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Suppose $\psi = \text{true} \in cl(\varphi)$.

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Suppose $\psi = \mathbf{true} \in cl(\varphi)$. Then $\mathbf{true} \in B_0$

note: \mathbf{true} is contained in all elementary formula-sets

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Suppose $\psi = \mathbf{true} \in cl(\varphi)$. Then $\mathbf{true} \in B_0$ and

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note: \mathbf{true} is contained in all elementary formula-sets
 \mathbf{true} holds for all paths/traces

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Suppose $\psi = \mathit{true} \in cl(\varphi)$. Then $\mathit{true} \in B_0$ and
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$$\mathbf{a} \in B_0$$

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Suppose $\psi = \mathbf{true} \in cl(\varphi)$. Then $\mathbf{true} \in B_0$ and

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Let $\psi = \mathbf{a} \in AP$. Then:

$$\mathbf{a} \in B_0 \iff \mathbf{a} \in A_0$$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

$$\forall F \in \mathcal{F} \exists j \geq 0. B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0 A_1 A_2 \dots \models \psi$$

Base of induction:

Suppose $\psi = \mathbf{true} \in cl(\varphi)$. Then $\mathbf{true} \in B_0$ and

$$A_0 A_1 A_2 \dots \models \mathbf{true}$$

Let $\psi = \mathbf{a} \in AP$. Then:

$$\mathbf{a} \in B_0 \iff \mathbf{a} \in A_0$$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

$$\forall F \in \mathcal{F} \exists j \geq 0. B_j \in F \quad A_0 = B_0 \cap AP$$

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Base of induction:

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Let $\psi = \mathbf{a} \in AP$. Then:

$$\mathbf{a} \in B_0 \iff \mathbf{a} \in A_0 \iff A_0 A_1 A_2 \dots \models \mathbf{a}$$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

$$\forall F \in \mathcal{F} \exists j \geq 0. B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

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Induction step: for $\psi = \neg\psi'$:

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

$$\forall F \in \mathcal{F} \exists j \geq 0. B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0 A_1 A_2 \dots \models \psi$$

Induction step: for $\psi = \neg\psi'$:

$$\psi \in B_0$$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

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$$\psi \in B_0 \quad \text{iff} \quad A_0 A_1 A_2 \dots \models \psi$$

Induction step: for $\psi = \neg\psi'$:

$$\psi \in B_0$$

$$\text{iff } \psi' \notin B_0 \quad (\text{maximal consistency})$$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

$$\forall F \in \mathcal{F} \exists j \geq 0. B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0 A_1 A_2 \dots \models \psi$$

Induction step: for $\psi = \neg\psi'$:

$$\psi \in B_0$$

iff $\psi' \notin B_0$ (maximal consistency)

iff $A_0 A_1 A_2 \dots \not\models \psi'$ (induction hypothesis)

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

$$\forall F \in \mathcal{F} \exists j \geq 0. B_j \in F$$

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$$\psi \in B_0$$

iff $\psi' \notin B_0$ (maximal consistency)

iff $A_0 A_1 A_2 \dots \not\models \psi'$ (induction hypothesis)

iff $A_0 A_1 A_2 \dots \models \psi$ (semantics of \neg)

$B \subseteq cl(\varphi)$ is elementary iff:

- (i) B is maximal consistent w.r.t. prop. logic, i.e., if $\psi, \psi_1 \wedge \psi_2 \in cl(\varphi)$ then:

$$\begin{aligned}\psi \notin B & \text{ iff } \neg\psi \in B \\ \psi_1 \wedge \psi_2 \in B & \text{ iff } \psi_1 \in B \text{ and } \psi_2 \in B \\ \text{true} \in cl(\varphi) & \text{ implies } \text{true} \in B\end{aligned}$$

- (ii) B is locally consistent with respect to until \mathbf{U} , i.e., if $\psi_1 \mathbf{U} \psi_2 \in cl(\varphi)$ then:

$$\begin{aligned}\text{if } \psi_1 \mathbf{U} \psi_2 \in B \text{ and } \psi_2 \notin B & \text{ then } \psi_1 \in B \\ \text{if } \psi_2 \in B & \text{ then } \psi_1 \mathbf{U} \psi_2 \in B\end{aligned}$$

$B \subseteq cl(\varphi)$ is elementary iff:

- (i) B is maximal consistent w.r.t. prop. logic, i.e., if $\psi, \psi_1 \wedge \psi_2 \in cl(\varphi)$ then:

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$$\psi_1 \wedge \psi_2 \in B \text{ iff } \psi_1 \in B \text{ and } \psi_2 \in B$$

$$true \in cl(\varphi) \text{ implies } true \in B$$

- (ii) B is locally consistent with respect to until \mathbf{U} , i.e., if $\psi_1 \mathbf{U} \psi_2 \in cl(\varphi)$ then:

$$\text{if } \psi_1 \mathbf{U} \psi_2 \in B \text{ and } \psi_2 \notin B \text{ then } \psi_1 \in B$$

$$\text{if } \psi_2 \in B \text{ then } \psi_1 \mathbf{U} \psi_2 \in B$$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

$$\forall F \in \mathcal{F} \exists j \geq 0. B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0 A_1 A_2 \dots \models \psi$$

Induction step: for $\psi = \psi_1 \wedge \psi_2$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

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Induction step: for $\psi = \psi_1 \wedge \psi_2$

$$\psi \in B_0$$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

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$$\psi \in B_0 \quad \text{iff} \quad A_0 A_1 A_2 \dots \models \psi$$

Induction step: for $\psi = \psi_1 \wedge \psi_2$

$$\psi \in B_0$$

$$\text{iff } \psi_1, \psi_2 \in B_0 \quad (\text{maximal consistency})$$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

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Induction step: for $\psi = \psi_1 \wedge \psi_2$

$$\psi \in B_0$$

iff $\psi_1, \psi_2 \in B_0$ (maximal consistency)

iff $A_0 A_1 A_2 \dots \models \psi_1$ and $A_0 A_1 A_2 \dots \models \psi_2$ (IH)

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

$$\forall F \in \mathcal{F} \exists j \geq 0. B_j \in F$$

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Induction step: for $\psi = \psi_1 \wedge \psi_2$

$$\psi \in B_0$$

iff $\psi_1, \psi_2 \in B_0$ (maximal consistency)

iff $A_0 A_1 A_2 \dots \models \psi_1$ and $A_0 A_1 A_2 \dots \models \psi_2$ (IH)

iff $A_0 A_1 A_2 \dots \models \psi$ (semantics of \wedge)

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

$$\forall F \in \mathcal{F} \exists j \geq 0. B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0 A_1 A_2 \dots \models \psi$$

Induction step: for $\psi = \bigcirc \psi'$:

$$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$$

state space: $Q = \{B \subseteq cl(\varphi) : B \text{ is elementary}\}$

initial states: $Q_0 = \{B \in Q : \varphi \in B\}$

transition relation: for $B \in Q$ and $A \in 2^{AP}$:

if $A \neq B \cap AP$ then $\delta(B, A) = \emptyset$

if $A = B \cap AP$ then $\delta(B, A) = \text{set of all } B' \in Q \text{ s.t.}$

$$\bigcirc \psi \in B \text{ iff } \psi \in B'$$

$$\psi_1 \mathbf{U} \psi_2 \in B \text{ iff } (\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \mathbf{U} \psi_2 \in B')$$

acceptance set $\mathcal{F} = \{F_{\psi_1 \mathbf{U} \psi_2} : \psi_1 \mathbf{U} \psi_2 \in cl(\varphi)\}$

where $F_{\psi_1 \mathbf{U} \psi_2} = \{B \in Q : \psi_1 \mathbf{U} \psi_2 \notin B \vee \psi_2 \in B\}$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

$$\forall F \in \mathcal{F} \exists j \geq 0. B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

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Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

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Induction step: for $\psi = \bigcirc \psi'$:

$$\psi \in B_0$$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

$$\forall F \in \mathcal{F} \quad \exists j \geq 0. B_j \in F \quad B_1 \in \delta(B_0, A_0)$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0 A_1 A_2 \dots \models \psi$$

Induction step: for $\psi = \bigcirc \psi'$:

$$\psi \in B_0$$

$$\text{iff } \psi' \in B_1 \quad (\text{definition of } \delta)$$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

$$\forall F \in \mathcal{F} \quad \exists j \geq 0. B_j \in F \quad B_1 \in \delta(B_0, A_0)$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0 A_1 A_2 \dots \models \psi$$

Induction step: for $\psi = \bigcirc \psi'$:

$$\psi \in B_0$$

$$\text{iff } \psi' \in B_1 \quad (\text{definition of } \delta)$$

$$\text{iff } A_1 A_2 A_3 \dots \models \psi' \quad (\text{induction hypothesis})$$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

$$\forall F \in \mathcal{F} \quad \exists j \geq 0. B_j \in F \quad B_1 \in \delta(B_0, A_0)$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0 A_1 A_2 \dots \models \psi$$

Induction step: for $\psi = \bigcirc \psi'$:

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$$\text{iff } \psi' \in B_1 \quad (\text{definition of } \delta)$$

$$\text{iff } A_1 A_2 A_3 \dots \models \psi' \quad (\text{induction hypothesis})$$

$$\text{iff } A_0 A_1 A_2 A_3 \dots \models \psi \quad (\text{semantics of } \bigcirc)$$

$B \subseteq cl(\varphi)$ is elementary iff:

- (i) B is maximal consistent w.r.t. prop. logic, i.e., if $\psi, \psi_1 \wedge \psi_2 \in cl(\varphi)$ then:

$$\begin{aligned} \psi \notin B & \quad \text{iff} \quad \neg\psi \in B \\ \psi_1 \wedge \psi_2 \in B & \quad \text{iff} \quad \psi_1 \in B \text{ and } \psi_2 \in B \\ \text{true} \in cl(\varphi) & \quad \text{implies} \quad \text{true} \in B \end{aligned}$$

- (ii) B is locally consistent with respect to until \mathbf{U} , i.e., if $\psi_1 \mathbf{U} \psi_2 \in cl(\varphi)$ then:

$$\begin{aligned} \text{if } \psi_1 \mathbf{U} \psi_2 \in B \text{ and } \psi_2 \notin B & \quad \text{then } \psi_1 \in B \\ \text{if } \psi_2 \in B & \quad \text{then } \psi_1 \mathbf{U} \psi_2 \in B \end{aligned}$$

$$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$$

state space: $Q = \{B \subseteq cl(\varphi) : B \text{ is elementary}\}$

initial states: $Q_0 = \{B \in Q : \varphi \in B\}$

transition relation: for $B \in Q$ and $A \in 2^{AP}$:

if $A \neq B \cap AP$ then $\delta(B, A) = \emptyset$

if $A = B \cap AP$ then $\delta(B, A) = \text{set of all } B' \in Q \text{ s.t.}$

$$\bigcirc \psi \in B \text{ iff } \psi \in B'$$

$$\psi_1 \mathbf{U} \psi_2 \in B \text{ iff } (\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \mathbf{U} \psi_2 \in B')$$

acceptance set $\mathcal{F} = \{F_{\psi_1 \mathbf{U} \psi_2} : \psi_1 \mathbf{U} \psi_2 \in cl(\varphi)\}$

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$$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$$

state space: $Q = \{B \subseteq cl(\varphi) : B \text{ is elementary}\}$

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acceptance set $\mathcal{F} = \{F_{\psi_1 \mathbf{U} \psi_2} : \psi_1 \mathbf{U} \psi_2 \in cl(\varphi)\}$

where $F_{\psi_1 \mathbf{U} \psi_2} = \{B \in Q : \psi_1 \mathbf{U} \psi_2 \notin B \vee \psi_2 \in B\}$

Induction step: until

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Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

$$\forall F \in \mathcal{F} \exists j \geq 0. B_j \in F$$

then for all $\psi \in cl(\varphi)$: $\psi \in B_0$ iff $A_0 A_1 A_2 \dots \models \psi$

Induction step for $\psi = \psi_1 \mathbf{U} \psi_2$:

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

$$\forall F \in \mathcal{F} \exists j \geq 0. B_j \in F$$

then for all $\psi \in cl(\varphi)$: $\psi \in B_0$ iff $A_0 A_1 A_2 \dots \models \psi$

Induction step for $\psi = \psi_1 \mathbf{U} \psi_2$:

“ \Leftarrow ”: Suppose $A_0 A_1 A_2 \dots \models \psi$.

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

$$\forall F \in \mathcal{F} \exists j \geq 0. B_j \in F$$

then for all $\psi \in cl(\varphi)$: $\psi \in B_0$ iff $A_0 A_1 A_2 \dots \models \psi$

Induction step for $\psi = \psi_1 \mathbf{U} \psi_2$:

“ \Leftarrow ”: Suppose $A_0 A_1 A_2 \dots \models \psi$. Let $j \geq 0$ s.t.

$$A_j A_{j+1} A_{j+2} \dots \models \psi_2$$

$$A_{j-1} A_j A_{j-1} \dots \models \psi_1$$

$$A_{j-2} A_{j-1} A_j \dots \models \psi_1$$

$$\vdots$$

$$A_0 A_1 A_2 A_3 \dots \models \psi_1$$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

$$\forall F \in \mathcal{F} \exists j \geq 0. B_j \in F$$

then for all $\psi \in cl(\varphi)$: $\psi \in B_0$ iff $A_0 A_1 A_2 \dots \models \psi$

Induction step for $\psi = \psi_1 \mathbf{U} \psi_2$:

“ \Leftarrow ”: Suppose $A_0 A_1 A_2 \dots \models \psi$. Let $j \geq 0$ s.t.

$$A_j A_{j+1} A_{j+2} \dots \models \psi_2 \stackrel{\text{IH}}{\Rightarrow} \psi_2 \in B_j$$

$$A_{j-1} A_j A_{j-1} \dots \models \psi_1 \Rightarrow \psi_1 \in B_{j-1}$$

$$A_{j-2} A_{j-1} A_j \dots \models \psi_1 \Rightarrow \psi_1 \in B_{j-2}$$

$$\vdots$$

$$A_0 A_1 A_2 A_3 \dots \models \psi_1 \Rightarrow \psi_1 \in B_0$$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

$\forall F \in \mathcal{F} \exists j \geq 0. B_j \in F$ B_j is elementary

then for all $\psi \in cl(\varphi)$: $\psi \in B_0$ iff $A_0 A_1 A_2 \dots \models \psi$

Induction step for $\psi = \psi_1 \mathbf{U} \psi_2$:

“ \Leftarrow ”: Suppose $A_0 A_1 A_2 \dots \models \psi$. Let $j \geq 0$ s.t.

$$\begin{array}{lclcl}
 A_j A_{j+1} A_{j+2} \dots & \models \psi_2 & \stackrel{\text{IH}}{\Rightarrow} & \psi_2 \in B_j & \Rightarrow \psi \in B_j \\
 A_{j-1} A_j A_{j-1} \dots & \models \psi_1 & \Rightarrow & \psi_1 \in B_{j-1} & \\
 A_{j-2} A_{j-1} A_j \dots & \models \psi_1 & \Rightarrow & \psi_1 \in B_{j-2} & \\
 \vdots & & & \vdots & \\
 A_0 A_1 A_2 A_3 \dots & \models \psi_1 & \Rightarrow & \psi_1 \in B_0 &
 \end{array}$$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

$$\forall F \in \mathcal{F} \exists j \geq 0. B_j \in F \quad B_j \in \delta(B_{j-1}, A_{j-1})$$

then for all $\psi \in cl(\varphi)$: $\psi \in B_0$ iff $A_0 A_1 A_2 \dots \models \psi$

Induction step for $\psi = \psi_1 \mathbf{U} \psi_2$:

“ \Leftarrow ”: Suppose $A_0 A_1 A_2 \dots \models \psi$. Let $j \geq 0$ s.t.

$$\begin{array}{l} A_j A_{j+1} A_{j+2} \dots \models \psi_2 \quad \stackrel{\text{IH}}{\Rightarrow} \quad \psi_2 \in B_j \quad \Rightarrow \quad \psi \in B_j \\ A_{j-1} A_j A_{j-1} \dots \models \psi_1 \quad \Rightarrow \quad \psi_1 \in B_{j-1} \quad \wedge \quad \psi \in B_{j-1} \\ A_{j-2} A_{j-1} A_j \dots \models \psi_1 \quad \Rightarrow \quad \psi_1 \in B_{j-2} \\ \vdots \\ A_0 A_1 A_2 A_3 \dots \models \psi_1 \quad \Rightarrow \quad \psi_1 \in B_0 \end{array}$$

Induction step: until (part “ \Leftarrow ”)

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Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

$$\forall F \in \mathcal{F} \exists j \geq 0. B_j \in F \quad B_{j-1} \in \delta(B_{j-2}, A_{j-2})$$

then for all $\psi \in cl(\varphi)$: $\psi \in B_0$ iff $A_0 A_1 A_2 \dots \models \psi$

Induction step for $\psi = \psi_1 \mathbf{U} \psi_2$:

“ \Leftarrow ”: Suppose $A_0 A_1 A_2 \dots \models \psi$. Let $j \geq 0$ s.t.

$$\begin{array}{lclcl}
A_j A_{j+1} A_{j+2} \dots \models \psi_2 & \stackrel{\text{IH}}{\Rightarrow} & \psi_2 \in B_j & \Rightarrow & \psi \in B_j \\
A_{j-1} A_j A_{j-1} \dots \models \psi_1 & \Rightarrow & \psi_1 \in B_{j-1} & \wedge & \psi \in B_{j-1} \\
A_{j-2} A_{j-1} A_j \dots \models \psi_1 & \Rightarrow & \psi_1 \in B_{j-2} & \wedge & \psi \in B_{j-2} \\
\vdots & & \vdots & & \\
A_0 A_1 A_2 A_3 \dots \models \psi_1 & \Rightarrow & \psi_1 \in B_0 & &
\end{array}$$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

$$\forall F \in \mathcal{F} \exists j \geq 0. B_j \in F \quad B_1 \in \delta(B_0, A_0)$$

then for all $\psi \in cl(\varphi)$: $\psi \in B_0$ iff $A_0 A_1 A_2 \dots \models \psi$

Induction step for $\psi = \psi_1 \mathbf{U} \psi_2$:

“ \Leftarrow ”: Suppose $A_0 A_1 A_2 \dots \models \psi$. Let $j \geq 0$ s.t.

$$\begin{array}{rclcl}
 A_j A_{j+1} A_{j+2} \dots & \models \psi_2 & \stackrel{\text{IH}}{\Rightarrow} & \psi_2 \in B_j & \Rightarrow \psi \in B_j \\
 A_{j-1} A_j A_{j-1} \dots & \models \psi_1 & \Rightarrow & \psi_1 \in B_{j-1} & \wedge \psi \in B_{j-1} \\
 A_{j-2} A_{j-1} A_j \dots & \models \psi_1 & \Rightarrow & \psi_1 \in B_{j-2} & \wedge \psi \in B_{j-2} \\
 & \vdots & & \vdots & \vdots \\
 A_0 A_1 A_2 A_3 \dots & \models \psi_1 & \Rightarrow & \psi_1 \in B_0 & \wedge \psi \in B_0
 \end{array}$$

Induction step: until (part “ \implies ”)

LTLMC3.2-64

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

$$\forall F \in \mathcal{F} \exists j \geq 0. B_j \in F$$

then for all $\psi \in cl(\varphi)$: $\psi \in B_0$ iff $A_0 A_1 A_2 \dots \models \psi$

Induction step for $\psi = \psi_1 \mathbf{U} \psi_2$:

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$$\forall F \in \mathcal{F} \quad \exists j \geq 0. B_j \in F \quad B_{i+1} \in \delta(B_i, A_i)$$

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$$\implies \psi \in B_2$$

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Contradiction!

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

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$$\neg \psi_2 \in B_{j-1}$$

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$$\vdots$$

$$\neg \psi_2 \in B_1$$

$$\neg \psi_2 \in B_0$$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

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$$\neg \psi_2, \psi \in B_0 \quad \longleftarrow \text{by assumption}$$

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\leftarrow local consistency w.r.t. \mathbf{U}

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in \mathcal{G} s.t.

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$$B_{i+1} \in \delta(B_i, A_i)$$

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 \vdots & \vdots & \vdots \\
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\Downarrow

$$A_0 A_1 A_2 \dots \models \psi = \psi_1 \mathbf{U} \psi_2$$

Complexity: LTL \rightsquigarrow NBA

LTLMC3.2-67

For each **LTL** formula φ , there is an **NBA** \mathcal{A} s.t.

$$\mathcal{L}_\omega(\mathcal{A}) = \text{Words}(\varphi)$$

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LTL formula φ

GNBA \mathcal{G}

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size: $\text{size}(\mathcal{G}) \cdot |\mathcal{F}|$

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LTL formula φ

GNBA \mathcal{G}

size: $2^{|\text{cl}(\varphi)|}$

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LTL formula φ

GNBA \mathcal{G}

size: $2^{|\text{cl}(\varphi)|}$

NBA \mathcal{A}

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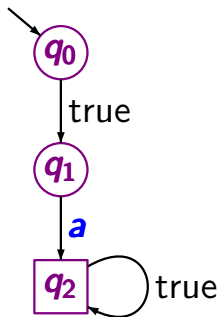
For the proposed transformation **LTL** \rightsquigarrow **NBA**:

The constructed NBA for LTL formulas are often
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unnecessarily complicated

NBA for $\bigcirc a$

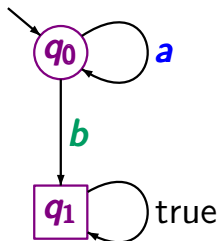


constructed GNBA has
4 states and **8** edges

For the proposed transformation **LTL** \rightsquigarrow **NBA**:

The constructed NBA for LTL formulas are often
unnecessarily complicated

NBA for **$aU b$**



constructed (G)NBA has
5 states and **20** edges

For the proposed transformation **LTL** \rightsquigarrow **NBA**:

The constructed NBA for LTL formulas are often
unnecessarily complicated

... but there exists LTL formulas φ_n such that

- $|\varphi_n| = \mathcal{O}(\text{poly}(n))$
- each NBA for φ_n has at least 2^n states

LT-properties that have no “small” NBA

LTLMC3.2-69

consider the following family of LT-properties $(E_n)_{n \geq 1}$:

$$E_n = \left\{ \begin{array}{l} \text{set of all infinite words over } 2^{AP} \text{ of the form} \\ A_1 A_2 A_3 \dots A_n A_1 A_2 A_3 \dots A_n B_1 B_2 B_3 B_4 \dots \end{array} \right.$$

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for some $\mathbf{x} \in (2^{AP})^*$
of length n arbitrary

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LTL formula φ_n with $Words(\varphi_n) = E_n$

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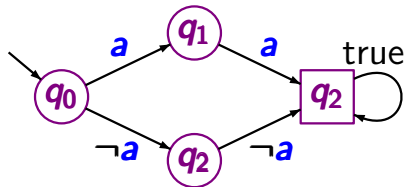
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NBA for E_1 if $AP = \{a\}$:

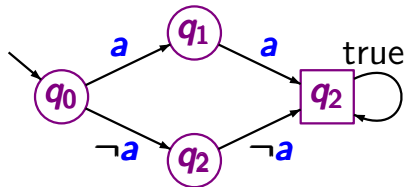


LT-property E_n for $n=1$

LTLMC3.2-69A

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LTL-formula:

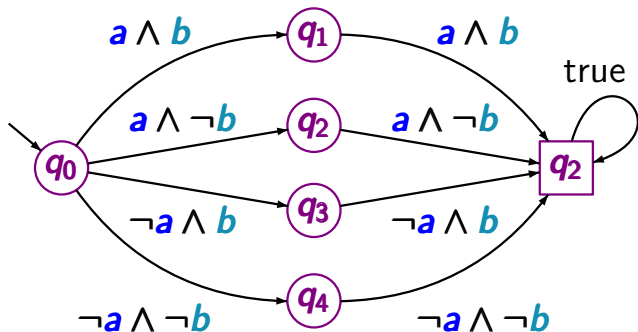
$$a \leftrightarrow \bigcirc a$$

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LTLMC3.2-69A

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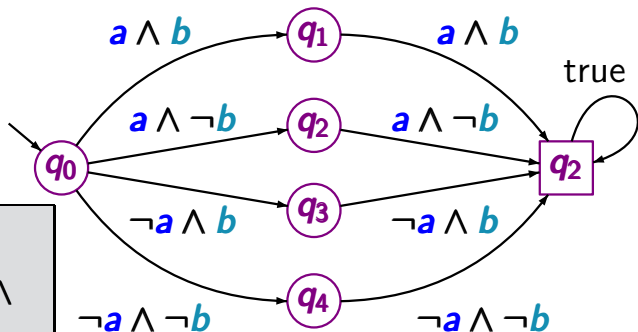


LT-property E_n for $n=1$

LTLMC3.2-69A

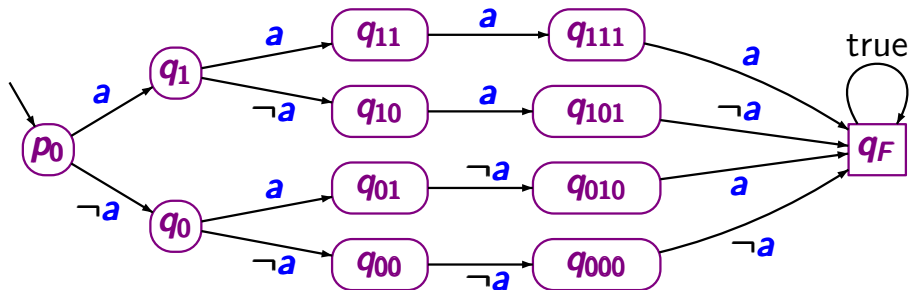
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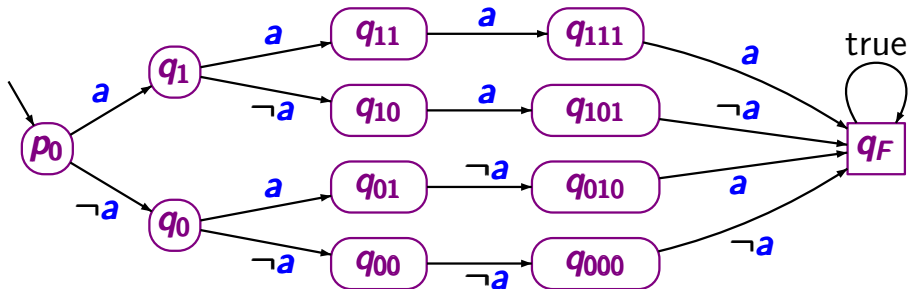


LTL-formula:

$$(a \leftrightarrow \bigcirc a) \wedge (b \leftrightarrow \bigcirc b)$$



$$E_2 = \{A_1 A_2 A_1 A_2 \sigma : A_1, A_2 \subseteq AP, \sigma \in (2^{AP})^\omega\}$$

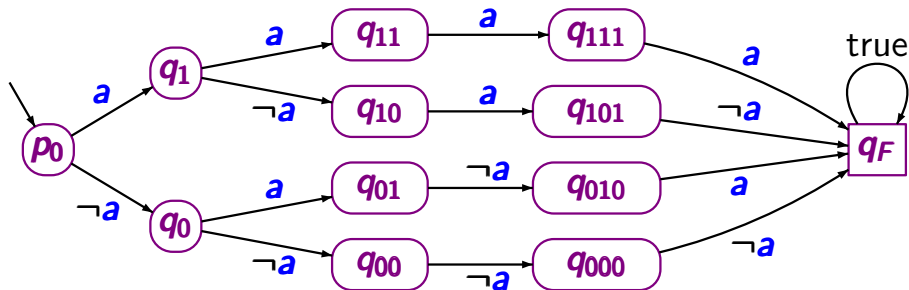


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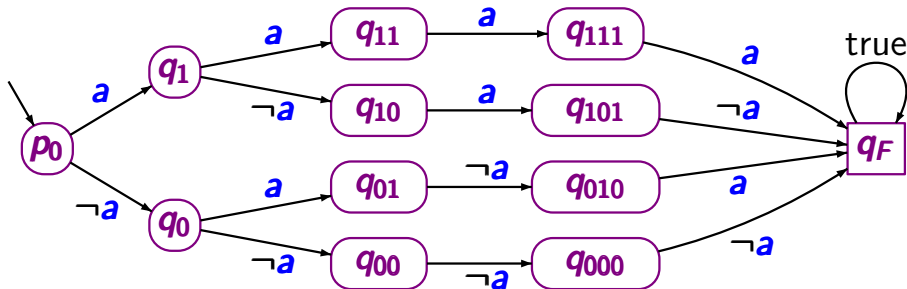
LTL-formula: $(a \leftrightarrow \bigcirc \bigcirc a) \wedge (\bigcirc a \leftrightarrow \bigcirc \bigcirc \bigcirc a)$

LT property E_n for $n=2$ and $AP = \{a\}$

LTLMC3.2-70



general case: each NBA for E_n has $\geq 2^n$ states

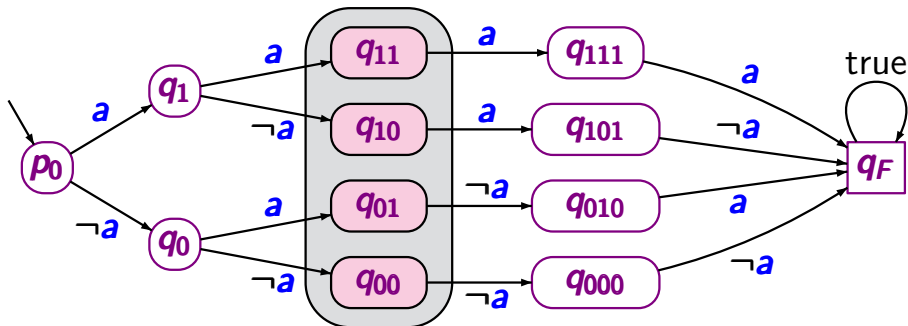


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LTLMC3.2-70



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